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Renormalization Group Analysis of Tachyon Condensation

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Abstract

Renormalization group analysis of boundary conformal field theory on bosonic D25-brane is used to study tachyon condensation. Placing the lump on a finite circle and triggering only the first three tachyon modes, the theory flows to nearby IR fixed point representing lumps that are extended object with definite profile. The boundary entropy corresponding to the D24-brane tension is calculated in the leading order in perturbative analysis which decreases under RG flow and agrees with the expected result to an accuracy of 8%. Multicritical behaviour of the IR theory suggests that the end point of the flow represents a configuration of two D24-branes. Analogy with Kondo physics is discussed.

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1 Introduction

It has been suggested by Sen [1] that the energy gap between unstable D-brane configurations and the stable vacuum should be computable using Witten's cubic string field theory (SFT) [2]. The level truncation scheme of [3] has appeared to lead to very good agreement with the expected results in the context of the decay of bosonic D-branes [4, 5] and unstable D-branes in superstring field theory [6, 7, 8, 9]. Lower dimensional D-branes have been constructed as tachyonic lump configurations on bosonic D25-brane [10, 11, 12, 13]. Support for Sen's conjecture that lower dimensional D-branes can be identified with tachyonic lump solution of string field theory on bosonic D25-brane also has come from the noncommutative limit of the effective field theory [14, 15, 16, 17]. Various toy models of tachyon dynamics have been useful tool for understanding the realization of Sen's conjectures [18, 19, 20].

Despite the success of cubic SFT in leading to very good agreement with the expected results, it is still not clear why the calculations in the level-truncated string field theory converge so rapidly to the correct answer for quantities like vacuum energy. It is also not clear how to study the nonperturbative vacuum using this approach. An alternative method has been suggested in [21] which says that renormalization group (RG) analysis of worldsheet theory in first quantized approach can be used to show that the mass of the tachyon lump on a Dp-brane is equal to that of a D(p–1)-brane. This leads to the idea that boundary string field theory (SFT) as was originally proposed by Witten and Shatashvili [22, 23, 24, 25] may efficiently describe open string tachyon condensation on D-branes in bosonic string theory [26, 27, 28]. It has been pointed out in [26, 27] that boundary SFT can provide an exact verification of Sen's conjectures. Based on [23, 25], one can compute the action exactly taking a simplest tachyon profile as boundary operator of ghost number zero which is quadratic in the space-time coordinate. One can also describe the lumps corresponding to the lower dimensional D-branes and calculate their tension that agrees with the expected result exactly.

One strong point of the results of [11] using level truncation scheme in cubic SFT is that it gives a definite picture of the tachyon profile as superposition of $\cos(\frac{n}{R}X)$ for different n with definite coefficients producing soliton of finite width. On the other hand the boundary SFT analysis of [27] does not say how the higher tachyon harmonics get mixed with the cosines of different n to produce finite size soliton profile. The reason for this is that the mass parameter flows to infinity in the IR in their particular choice of coordinates starting with the simple choice of the tachyon profile initially. This is because the RG equations in that particular choice of coordinates become linear in coupling constants. Since the width of the soliton is given by inverse of the mass parameter which flows to infinity in the exact description, the width is zero. Also it is difficult to see in this setup how the theory at IR fixed point can describe configurations of more than one D-brane. We should note that although the theory flows to infinity in the IR, the Zamolodchikov metric on the space of worldsheet theories measures the distance between the UV and IR fixed point to be finite.

In this paper we will try to address these issues in the worldsheet approach by choosing the initial tachyon profile on a circle on bosonic D25-brane world-volume. The basic setup is similar to [11]. The choice of boundary perturbation, as we will discuss in the next section, is motivated from the analysis of [11]. In our choice of coordinates in the space of worldsheet field theories the RG analysis provides the boundary conformal field theory to flow to a nearby IR fixed point. We will consider only the first three tachyon modes. The values they flow to after RG flow appears to be in good agreement with the values computed in [11] of tachyon modes of the string field at the stationary point of the potential.

The plan of the paper is the following. In section 2 we briefly review the worldsheet RG scheme in the context of boundary SFT. We comment on the boundary entropy that measures the corresponding D-brane tension describing the boundary CFT. We highlight the basic setup of [11] that leads to a particular choice of the tachyon profile.

In section 3 we perform the RG analysis in detail choosing a particular ansatz for the Green's function of fast moving modes of the scalar field. The RG equations are obtained up to third order in coupling constants. The easy but tedious parts of the calculations are given in the appendix.

In section 4 we plot the lump profile that appears from the RG analysis of the previous section. Following the method of Affleck and Ludwig [29] we calculate the boundary entropy in the leading order. Our result satisfies the g-theorem of [29]. The boundary entropy is calculated to an accuracy of about 8% of the exact result.

In section 5 we analyse the multicritical behaviour of the potential in the IR and argue that the theory in the IR is that of a configuration of two D24-branes. We make an analogy with the Kondo problem and argue that the process of the formation of the lump due to tachyon condensation corresponds to the underscreened Kondo effect. Exact screening occurs when the theory on the brane rolls down to the nonperturbative closed string vacuum. Overscreening, on the other hand leads to a picture similar to dielectric effect. We comment on $U(\infty)$ symmetry restoration.

Section 6 contains discussions on further related issues that are beyond the scope of the paper.

2 Open string in tachyon background

If the spectrum of a point in the moduli space has relevant operators in the IR, the point is said to be unstable. The unstable point then might be discarded from the moduli space. In other words, this results in the appearance of unstable directions in the effective potential. IR instability indicates that we are in a *false* vacuum. The obvious question will be which point (or vacuum) will replace this unstable point, or in other words, how to resolve the IR instability.

Often regions in moduli space contain D-brane configurations related by T-duality. The above issue can be addressed in these sectors by adding some IR relevant boundary perturbations. One important feature of this boundary deformation is that the bulk theory always remain conformal. Flows caused by a relevant boundary operator appear as open string tachyon condensation. Flowing to IR on the worldsheet is equivalent to approaching a classical solution of spacetime theory.

As a result of the flow, at some points in moduli space, boundary conditions are changed from Neumann to Dirichlet. The reverse process indicates nonunitarity. The direction of the flow is determined by *boundary entropy* defined by [30]

$$g_a = \langle 0|a\rangle \,, \tag{2.1}$$

the disk partition function of the boundary state $|a\rangle$ associated to the perturbed theory. The phases of $|0\rangle$ and $|a\rangle$ can be chosen such that the above quantity is real and positive for any boundary state. It is shown in [29] in first order in conformal perturbation theory that boundary entropy, g decreases along RG flows, suggesting the so called *g-theorem* similar to Zamolodchikov's c-theorem [31]. In this context, the case of Sine-Gordon boundary perturbation with single frequency is studied in [32, 33].

In fact g measures the tension of the D-brane that describes boundary conformal field theory at the corresponding fixed point [34] (see also [35]) and the g-theorem implies the minimization of the action in the space of *all two-dimensioanal worldsheet field theories*. The worldsheet formulation of string field theory is *manifestly* background independent[1]. In [22] a gauge invariant background independent spacetime string field theory action is defined as the solution of the following equation:

$$\frac{\partial \mathcal{S}_{\rm bsft}}{\partial \lambda_i} = \frac{K}{2}\int_0^{2\pi}\frac{d\theta}{2\pi}\int_0^{2\pi}\frac{d\theta'}{2\pi}\,\langle \mathcal{O}_i(\theta)\{Q,\mathcal{O}(\theta')\}\rangle_\lambda \,, \tag{2.2}$$

where Q is the BRST charge and and θ is the boundary parameter of the disk. The operator $\mathcal{O}$ has ghost number one: $\mathcal{O} = c\mathcal{V}$, where $\mathcal{V}$ is a general boundary perturbation describing the

[1]Of course, here we are referring to the open string background. The definition of Q depends on the closed string background.

space of boundary perturbations: $\mathcal{V} = \sum_i \lambda^i \mathcal{V}_i$. The normalization constant K is fixed in [28] by comparing the on-shell three tachyon amplitude computed from the background independent open string field theory with the same computed in Chern-Simons string field theory:

$$K = T_p \,, \tag{2.3}$$

T_p being the corresponding Dp-brane tension. The correlation function in the above expression is defined in terms of the perturbed worldsheet action on the disk. The above action is defined up to an additive constant. Although the boundary perturbation $\mathcal{V}$ does not depend on ghosts, the theory with perturbed action is not renormalizable. But in the case of tachyon condensation, it is renormalizable. So boundary string field theory works well for studying tachyon condensation problem. But it is not clear how to define a general off-shell amplitude due to UV divergences on the worldsheet.

An alternative definition of S is given by the metric on the space of worldsheet theories [27]

$$\frac{\partial \mathcal{S}_{\text{bsft}}}{\partial \lambda_i} = -\beta^j G_{ij} \,, \tag{2.4}$$

where β_i is the corresponding beta function which acts like a vector field on the space of worldsheet theories. In [23, 25] a very important relation between the action $\mathcal{S}$ and the partition function on the disk is demonstrated up to second order in coupling constant:

$$\mathcal{S}_{\text{bsft}} = -\beta^i \frac{\partial Z(\lambda)}{\partial \lambda^i} + Z(\lambda) \,. \tag{2.5}$$

Since on-shell the beta function vanishes, the action is same as the partition function. This fixes the additive ambiguity in (2.2). The above relation implies that all symmetries of the worldsheet partition partition function are also symmetries of the spacetime action. It is further argued that [27] the spacetime action $\mathcal{S}$ is nothing but the boundary entropy g. Although background independence is manifest in the worldsheet formalism, it is lost once we compute $\mathcal{S}$ or g perturbatively. But if the above relation between the spacetime action and the partition function is true in all orders in coupling constant, it is not lost.

On the other hand, the Chern-Simons string field theory is not manifestly background independent. To achieve this one is forced to work in a truncated version of Hilbert space of the first quantized theory restricting the string field to a background independent subspace for studying the classical lump solution. In our renormalization group analysis of the problem of formation of the tachyonic lump, we follow the basic setup of [11] where the x_{25} coordinate is taken to be on a circle of radius R_{25}. We take Φ to be the scalar field on the string worldsheet associated with x_{25}. On a D25-brane Φ satisfies Neumann boundary conditions. The conformal field theory associated with the field Φ has central charge, $c = 1$.

Before going into the RG analysis let us briefly discuss the setup of [11]. For details on the background independent subspace of the complete Fock space see [1]. A basis of states of this theory is conveniently formed by grouping the states into Verma modules. The states contained in the Verma module are obtained out of primary state $\exp(in\Phi(0)/R_{25})|0\rangle$ by acting with the associated Virasoro generators L^{Φ}_{-m}. Linear independence of the states to form the basis is achieved by removing null states from the spectrum. In fact the spectrum is free of null states for $n \neq 0$ and an irrational R_{25} value. In order to achieve a successful truncation of the Hilbert space one has to restrict the primary states of the boundary conformal field theory along x_{25} to be even under $\Phi \to -\Phi$ and to be trivial primaries of the conformal field theory of the fields associated to the rest of the coordinates of bosonic string theory. As a result the appropriate primary states are: (1) the zero momentum primaries that are even under $\Phi \to -\Phi$ (also removing the null states), and (2) the vacuum state $\cos(n\Phi(0)/R_{25})|0\rangle$ with $n \neq 0$.

Let us recall that an open bosonic string propagating in the tachyon background is described by the following action

$$S = \frac{1}{4\pi}\int dsdt\ \eta^{ab}\partial_a\Phi_\mu\partial_b\Phi^\mu + \int_{-\infty}^{\infty}\frac{dt}{a}\int dk\ T(k)e^{ik\Phi}\,, \tag{2.6}$$

where a is the UV cutoff and $T(k)$ is the tachyon field with momentum k. In the present case where x_{25} is compact, the momentum is descrete and we are motivated to take the boundary perturbation to be of the form (respecting $\Phi \to -\Phi$ symmetry):

$$\int dt\ T[\Phi(t)] = \sum_{n=0}^{\infty}\lambda_n\int dt\ \cos\Big(\frac{n}{R_{25}}\Phi(0,t)\Big)\,, \tag{2.7}$$

where λ_ns are tachyon modes. The zero mode λ_0 is the identity operator and is always tachyonic. The higher modes may not be tachyonic depending on the radius R_{25}. In our analysis we only consider the first three tachyon modes (*i.e.* $n = 0, 1, 2$). Hence the problem is similar to the boundary Sine-Gordon model with two frequencies, which typically allows to hit a nearby IR fixed point producing a lump profile of finite size.

3 Switching on first three tachyon modes: RG analysis

The boundary conformal field theory can be described by a Gaussian model with gapless excitation spectrum and the correlation functions of bosonic exponents follow power laws. This behaviour implies that the correlation length is infinite and the system is in its critical phase. Under boundary perturbations, usually the correlation functions are affected differently on different scales and the long distance asymptotics get affected the most. Certain perturbations may cause only tiny changes in the UV, but changes the IR behaviour profoundly. In the RG picture this is observed as a growth of the coupling constant associated with the perturbation. A slow decay of correlation functions gives rise to divergences in the perturbation series.

If the influence of the perturbing operator grows on large scales, the perturbation is *relevant*. Consider the perturbed action

$$S = S_0 + \lambda \int dt \mathcal{O}_\Delta(t) \tag{3.1}$$

where S_0 is the action of the system at criticality and $\mathcal{O}_\Delta(t)$ is the perturbing field with scaling dimension Δ, $\langle \mathcal{O}_\Delta(t_1)\mathcal{O}^+_\Delta(t_2)\rangle \sim |t_1 - t_2|^{-2\Delta}$. The perturbation with zero conformal spin and scaling dimension Δ is relevant if $\Delta < 1$ and *irrelevant* if $\Delta > 1$. The case $\Delta = 1$ is the *marginal* one and its effect on scaling dimensions of correlation functions depends on the sign of the coupling constant λ.

Consider the path integral on the disk with a worldsheet action

$$S = S_0 + \int dt\ T[\Phi(t)] \tag{3.2}$$

where S_0 is the free field action on the disk describing an open and closed string conformal background. The perturbing boundary field $T[\Phi(t)]$ is a tachyon profile and Φ is the scalar field along the string worldsheet associated with the compactified coordinate x_{25}. The above action describes a renormalization group flow from a theory where all 26 bosonic directions are Neumann describing a D25-brane to the theory where x_{25} direction is Dirichlet describing a D24-brane.

Instead of using the Callan-Symanzik formalism we follow the Wilsonian renormalization scheme where we put the theory on a lattice. We assume that the Fourier transform of the field, $\Phi(\omega)$, is defined in the Brillouin zone and choose a cut-off, $\omega < \Lambda$. The aim is to start moving towards larger distances (or lower energies) by integrating out the fields with shorter and shorter wavelengths. The procedure is based on the decomposition of the boundary field with a cutoff into a slow moving (long-wavelength) and a fast moving (short-moving) component, $\Phi_\Lambda = \Phi_{s\Lambda'} + \Phi_f$, where we have split the Brillouin zone $\omega < \Lambda$ into a wide region $0 < \omega < \Lambda' = \Lambda - d\Lambda$ and a narrow slice $\Lambda' = \Lambda - d\Lambda < \omega < \Lambda$. The original field is given by

$$\begin{aligned} \Phi_\Lambda(t) &= \int \frac{d\omega}{2\pi} e^{i\omega t}\Phi_\Lambda(\omega) \\ &= \int_{|\omega|<\Lambda'} \frac{d\omega}{2\pi} e^{i\omega t}\Phi_{s\Lambda'}(\omega) + \int_{\Lambda'<|\omega|<\Lambda} \frac{d\omega}{2\pi} e^{i\omega t}\Phi_f(\omega)\,. \end{aligned} \tag{3.3}$$

The next step will be to perform a partial path integration in the partition function over the fast moving part and representing the result in terms of an effective model for the slow moving field. If the model is renormalizable, which is known to be the case for Sine-Gordon model that we will consider below, the effective action will have the same structure as the original one, but with a new set of coupling constants. This procedure is repeated several times, and after each step the form of the original model is reproduced (upto irrelevant terms). The relations between the bare and the renormalized couplings then lead to renormalization group (RG) equations.

The Gaussian part S_0 is additive under the above decomposition. Hence, given the cutoff, the partition function is given by

$$\begin{aligned} Z_\Lambda &= \int D\Phi_{s\Lambda'} D\Phi_f \; e^{-S_0[\Phi_{s\Lambda'}]} e^{-S_0[\Phi_f]} e^{-S_I[\Phi_{s\Lambda'}+\Phi_f]} \\ &= Z_f \int D\Phi_{s\Lambda'} \; e^{-S_0[\Phi_{s\Lambda'}]} \langle e^{-S_I[\Phi_{s\Lambda'}+\Phi_f]} \rangle_f \end{aligned} \tag{3.4}$$

where Z_f is a nonsingular contribution of fast moving components to the partition function. The effective action involving the slow moving part of the field is given by

$$S_{eff}[\Phi_{s\Lambda'}] = S_0[\Phi_{s\Lambda'}] + S_{I\ eff}[\Phi_{s\Lambda'}] \tag{3.5}$$

where

$$S_{I\ eff}[\Phi_{s\Lambda'}] = -\ln\langle e^{-S_I[\Phi_{s\Lambda'}+\Phi_f]} \rangle_f \tag{3.6}$$

It is clear that the effective action preserves all IR singularities. Assuming the coupling to be small, we expand the above effective interaction perturbatively upto cubic order

$$\begin{aligned} S_{I\ eff}[\Phi_{s\Lambda'}] &= \langle S_I \rangle_f + \frac{1}{2}(\langle S_I \rangle_f^2 - \langle S_I^2 \rangle_f) \\ &\quad + (\frac{1}{6}\langle S_I^3 \rangle - \frac{1}{2}\langle S_I \rangle_f \langle S_I^2 \rangle_f + \frac{1}{3}\langle S_I \rangle_f^3) + O(\lambda^4) \end{aligned} \tag{3.7}$$

Now we consider the effect of switching on first three tachyon harmonics: λ_0, λ_1, and λ_2. The scalar perturbation is of *double Sine-Gordon* type and reads [2]

$$S_I[\Phi_\Lambda] = \lambda_0 \int dt\ \mathbf{1} - \lambda_1 \int dt\ \cos\beta\Phi_\Lambda(t) - \lambda_2 \int dt\ \cos 2\beta\Phi_\Lambda(t)\,. \tag{3.8}$$

[2] One can take $(\pm 1)^n$ symmetry of the coupling λ_ns in writing down the perturbation which might be more appropriate in order to compare the results with that of [11]. However we chose perturbation of the type (3.8) as this allows us to hit the desired multicritical IR fixed point from the stable direction. However we will invoke the $(-1)^n$ symmetry to show multicriticality in section 5. The opposite sign in front of the identity operator does not modify the analysis for other operators. The coupling in front of the identity operator appears linearly in the beta function and it does not mix with other couplings. Also as we will see in the next section, boundary entropy does not get a contribution from it.

where β defines the conformal dimensions of our theory and the scaling dimension is $\Delta_1 = \beta^2/4\pi$ for the first harmonic and $\Delta_2 = \beta^2/\pi$ for the second harmonic. The tachyon zero mode is the identity operator with zero scaling dimension. Rescaling β: $\beta \to \sqrt{4\pi}\beta$ results in the relation $\Delta_n = n/R_{25}$. Following [11], we take the radius $R_{25} = \sqrt{3}$. In this radius the first mode is least relevant and the perturbation by the second mode is least irrelevant, which, after being added to the cutoff theory, should improve the RG result. Such a procedure should lead to a good result even when only first few irrelevant terms are included. The effects of highly irrelevant operators are highly damped by their rapid decay.

Using the above definition of the effective action we perturbatively calculate contributions in each order in coupling constant, λ. Then in order to restore the original cut-off to Λ we rescale the energy $\omega' = (\Lambda/\Lambda')\omega \simeq (1+dl)\omega$ and the time $t' = (1-dl)t$ so that the effective action is of the same form as the bare one but with a renormalized strength of coupling.

Now we calculate the expression (3.7) term by term.

<u>*First order contribution*</u>:

In first order in couplings, it is given by

$$\begin{aligned} \langle S_I \rangle_f &= \lambda_0 \int dt\, \langle \mathbf{1} \rangle_f - \lambda_1 \int dt\, \langle \cos\beta[\Phi_{s\Lambda'}(t) + \Phi_f(t)] \rangle_f \\ &\quad - \lambda_2 \int dt\, \langle \cos 2\beta[\Phi_{s\Lambda'}(t) + \Phi_f(t)] \rangle_f \,. \end{aligned} \tag{3.9}$$

The expression for correlation function of bosonic exponents reads

$$\langle \prod_n e^{i\beta_n \Phi(t_n)} \rangle = e^{-\sum_{m>n} \beta_m \beta_n G(t_m,t_n)} e^{-\frac{1}{2}\sum_m \beta_n^2 G(t_m,t_m)} \,. \tag{3.10}$$

The terms containing Green's functions of coinciding arguments are singular in the continuous limit. But in our regularized theory they are finite. The Green's function has the following well known form

$$G(t_m, t_n) = \frac{1}{4\pi} ln\left(\frac{R^2}{(t_m - t_n)^2 + a^2} \right), \tag{3.11}$$

where R and a are IR and UV cut-off respectively. Substituting into (3.10) we get the result for the correlation function of bosonic exponents:

$$\langle \prod_n e^{i\beta_n \Phi(t_n)} \rangle = \prod_{m>n} \left(1 + \frac{(t_m - t_n)^2}{a^2} \right)^{\frac{\beta_m \beta_n}{4\pi}} \left(\frac{R}{a} \right)^{-\frac{\left(\sum_n \beta_n\right)^2}{4\pi}} \,. \tag{3.12}$$

Using the above formulae we obtain

$$\langle e^{\pm i\beta\phi_f(t)}\rangle_f = e^{-\frac{\beta^2}{2}\langle\Phi_f(0)^2\rangle_f} = e^{-\frac{\beta^2}{2}G_f(0,0)}\,, \tag{3.13}$$

where

$$-\frac{\beta^2}{2}G(0,0) = -\frac{\beta^2}{2}\int_0^\Lambda d\omega f(\omega) = -\frac{\beta^2}{4\pi}\; ln\Lambda\,, \tag{3.14}$$

and

$$-\frac{\beta^2}{2}G_f(0,0) = -\frac{\beta^2}{2}\int_{\Lambda-d\Lambda}^{\Lambda} d\omega f(\omega) = d\Lambda I'(\Lambda) = -\frac{\beta^2}{4\pi}\Big(\frac{d\Lambda}{\Lambda}\Big) = -\frac{\beta^2}{4\pi}dl\,. \tag{3.15}$$

Hence (3.9) turns out to be

$$\begin{aligned}\langle S_I\rangle_f &= \tilde{\lambda}_0\int\frac{dt}{a}\,\mathbf{1} - \tilde{\lambda}_1\Big(1-\frac{\beta^2}{4\pi}dl\Big)\int\frac{dt}{a}\;\cos\beta\Phi_{s\Lambda'}(t)\\ &\quad -\tilde{\lambda}_2\Big(1-\frac{4\beta^2}{4\pi}dl\Big)\int\frac{dt}{a}\;\cos 2\beta\Phi_{s\Lambda'}(t) + O(dl^2)\,,\end{aligned} \tag{3.16}$$

where $\tilde{\lambda}_i = \lambda_i a$ are small dimensionless couplings.

<u>*Second order contribution*</u>:

We now turn to the quadratic contribution to (3.7). Imposing translational invariance it can be written as

$$\begin{aligned}&\frac{1}{2}(\langle S_I\rangle_f^2 - \langle S_I^2\rangle_f)\\ =\;& \frac{\lambda_1^2}{4}\int dt_1\int dt_2\Big\{e^{-\beta^2 G_f(0,0)}\Big(1-e^{-\beta^2 G_f(t_2-t_2)}\Big)\cos\beta[\Phi_{s\Lambda'}(t_1)+\Phi_{s\Lambda'}(t_2)]\\ &+e^{-\beta^2 G_f(0,0)}\Big(1-e^{\beta^2 G_f(t_2-t_2)}\Big)\cos\beta[\Phi_{s\Lambda'}(t_1)-\Phi_{s\Lambda'}(t_2)]\Big\}\\ &+\frac{\lambda_2^2}{4}\int dt_1\int dt_2\Big\{e^{-4\beta^2 G_f(0,0)}\Big(1-e^{-4\beta^2 G_f(t_2-t_2)}\Big)\cos 2\beta[\Phi_{s\Lambda'}(t_1)+\Phi_{s\Lambda'}(t_2)]\\ &+e^{-4\beta^2 G_f(0,0)}\Big(1-e^{4\beta^2 G_f(t_2-t_2)}\Big)\cos 2\beta[\Phi_{s\Lambda'}(t_1)-\Phi_{s\Lambda'}(t_2)]\Big\}\\ &+\frac{\lambda_1\lambda_2}{2}e^{-\frac{5\beta^2}{4\pi}dl}\int dt_1\int dt_2\;\Big\{(1-e^{-2\beta^2 G_f(t_1-t_2)})\cos\beta[\Phi_{s\Lambda'}(t_1)+2\Phi_{s\Lambda'}(t_2)]\\ &+\{(1-e^{2\beta^2 G_f(t_1-t_2)})\cos\beta[2\Phi_{s\Lambda'}(t_2)-\Phi_{s\Lambda'}(t_1)]\Big\}\,.\end{aligned} \tag{3.17}$$

To evaluate the correlation functions of the bosonic vertices we have used (3.10). To evaluate the Green's function for fast moving modes we use the following scheme. Instead of considering just the momentum shell for fast moving modes, if we consider all momenta upto cutoff Λ then using (3.12) we get

$$\left\langle e^{i\sigma\beta\Phi(t_1)} \cdot e^{i\sigma\beta\Phi(t_2)}\right\rangle = \Big(1 + \frac{(t_1-t_2)^2}{a^2}\Big)^{\frac{\beta^2}{4\pi}} \Lambda^{-\frac{\beta^2}{\pi}}\,, \tag{3.18}$$

which, like (3.14), leads to

$$-\beta^2 G(0,0) - \beta^2 G(t_1-t_2) = \frac{\beta^2}{4\pi}\Big[ln\Big(1+\frac{(t_1-t_2)^2}{a^2}\Big) - ln\Lambda^4\Big]\,. \tag{3.19}$$

Hence the analogous treatment for fast moving modes follows by performing the integration in the narrow slice only. Using the expression for $G_f(0,0)$ given by (3.15) we arrive at the following useful expression for fast moving components

$$-\beta^2 G_f(t_1-t_2) \approx -\frac{\beta^2}{2\pi}dl\Big(ln\Big|\frac{a}{t_1-t_2}\Big| + 1\Big) + O(dl^2)\,, \tag{3.20}$$

where we have assumed that $|t_1 - t_2| \gg a$. It is clear that

$$G_f(t_1-t_2) = F(r)\ dl + O(dl^2)\,, \tag{3.21}$$

where $r = |t_1 - t_2|$. If we adopt a sharp momentum cut-off prescription, $F(r)$ will be Bessel function of order zero, $F(x) = (1/2\pi)J_0(\Lambda|x|)$, which has a long oscillating tail and does not fall off rapidly on increasing its argument. However, as was shown in [36, 37], in a smooth cut-off procedure $F(r)$ is truly short-ranged, essentially nonzero at $r < \Lambda^{-1} \sim a$. This can be seen in (3.17), the functions like $1 - e^{\pm\beta^2 G_f(t_2-t_2)}$ are also short-ranged. This allows us to introduce the center-of-mass coordinate $\tilde{R} = (t_1+t_2)/2$ and relative coordinate $r = t_1 - t_2$ and expand the cosines of (3.17) in r.

The detail calculation is shown in the appendix. Here we only mention some of the necessary facts. For example, consider the cosine $\cos\beta[\Phi_{s\Lambda'}(t_1) - \Phi_{s\Lambda'}(t_2)] \approx 1 - \frac{\beta^2 r^2}{2}(\partial_{\tilde{R}}\Phi_{s\Lambda'}(\tilde{R}))^2$, where the first term contributes to renormalization of the free energy. In our case that generates the RG contribution to identity operator. On the other hand, the second (gradient)^2 term is an irrelevant term with a factor proportional to the UV cut-off a in front of the renormalized coupling. Here we see a striking difference between bulk and boundary RG flows, where in the latter case the gradient term is responsible for renormalization of the constant β. Collecting the $O(\tilde{\lambda}_1^2)$, $O(\tilde{\lambda}_2^2)$, and $O(\tilde{\lambda}_1\tilde{\lambda}_2)$ contributions from the appendix, we arrive at the following complete second order expression

$$\begin{aligned} &\frac{1}{2}(\langle S_I\rangle_f^2 - \langle S_I^2\rangle_f) \\ = \ &\tilde{\lambda}_1^2\frac{\beta^2}{2\pi}dl\int\frac{dt}{a}\ \cos 2\beta\Phi_{s\Lambda'}(t) - \tilde{\lambda}_1^2\frac{\beta^2}{2\pi}dl\int\frac{dt}{a}\ \mathbf{1}\end{aligned}$$

$$
\begin{aligned}
&+\tilde{\lambda}_2^2\frac{2\beta^2}{\pi}dl\int\frac{dt}{a}\ \cos 4\beta\Phi_{s\Lambda'}(t)-\tilde{\lambda}_2^2\frac{2\beta^2}{\pi}dl\int\frac{dt}{a}\ \mathbf{1}\\
&+2\frac{\tilde{\lambda}_1\tilde{\lambda}_2}{\pi}\beta^2 dl\int\frac{d\tilde{R}}{a}\ (\cos 3\beta\Phi_{s\Lambda'}(\tilde{R})-\cos\beta\Phi_{s\Lambda'}(\tilde{R}))\,. \qquad (3.22)
\end{aligned}
$$

<u>*Third order contribution*</u>:

The calculation of the cubic contribution to (3.7) is similar and the details are given in the appendix. Here we give the result only:

$$
\begin{aligned}
&\frac{1}{6}\langle S_I^3\rangle_f-\frac{1}{2}\langle S_I\rangle_f\langle S_I^2\rangle_f+\frac{1}{3}\langle S_I\rangle_f^3\\
=\ &\tilde{\lambda}_1^3\Big(\frac{\pi\beta^2}{144}\Big)dl\int\frac{dt}{a}\ \cos 3\beta\Phi_{s\Lambda'}(t)+\tilde{\lambda}_1^3\Big(\frac{\pi\beta^2}{144}\Big)dl\int\frac{dt}{a}\ \cos\beta\Phi_{s\Lambda'}(t)\\
&+\tilde{\lambda}_2^3\Big(\frac{\pi\beta^2}{36}\Big)dl\int\frac{dt}{a}\ \cos 6\beta\Phi_{s\Lambda'}(t)+\tilde{\lambda}_2^3\Big(\frac{\pi\beta^2}{36}\Big)dl\int\frac{dt}{a}\ \cos 2\beta\Phi_{s\Lambda'}(t)\,. \qquad (3.23)
\end{aligned}
$$

Collecting all the results up to third order from (3.16), (3.22) and (3.23) we express the effective boundary action as

$$
\begin{aligned}
S_{I\ eff}[\Phi_{s\Lambda'}]\ =\ &\tilde{\lambda}_0\int\frac{dt}{a}\ \mathbf{1}-\tilde{\lambda}_1\Big(1-\frac{\beta^2}{4\pi}dl\Big)\int\frac{dt}{a}\ \cos\beta\Phi_{s\Lambda'}(t)-\tilde{\lambda}_2\Big(1-\frac{\beta^2}{\pi}dl\Big)\int\frac{dt}{a}\ \cos 2\beta\Phi_{s\Lambda'}(t)\\
&+\tilde{\lambda}_1^2\frac{\beta^2}{2\pi}dl\int\frac{dt}{a}\ \cos 2\beta\Phi_{s\Lambda'}(t)-\tilde{\lambda}_1^2\frac{\beta^2}{2\pi}dl\int\frac{dt}{a}\ \mathbf{1}\\
&+\tilde{\lambda}_2^2\frac{2\beta^2}{\pi}dl\int\frac{dt}{a}\ \cos 4\beta\Phi_{s\Lambda'}(t)-\tilde{\lambda}_2^2\frac{2\beta^2}{\pi}dl\int\frac{dt}{a}\ \mathbf{1}\\
&+2\frac{\tilde{\lambda}_1\tilde{\lambda}_2}{\pi}\beta^2 dl\int\frac{dt}{a}\ \cos 3\beta\Phi_{s\Lambda'}(\tilde{R})-2\frac{\tilde{\lambda}_1\tilde{\lambda}_2}{\pi}\beta^2 dl\int\frac{dt}{a}\ \cos\beta\Phi_{s\Lambda'}(\tilde{R})\\
&\tilde{\lambda}_1^3\Big(\frac{\pi\beta^2}{144}\Big)dl\int\frac{dt}{a}\ \cos 3\beta\Phi_{s\Lambda'}(t)+\tilde{\lambda}_1^3\Big(\frac{\pi\beta^2}{144}\Big)dl\int\frac{dt}{a}\ \cos\beta\Phi_{s\Lambda'}(t)\\
&+\tilde{\lambda}_2^3\Big(\frac{\pi\beta^2}{36}\Big)dl\int\frac{dt}{a}\ \cos 6\beta\Phi_{s\Lambda'}(t)+\tilde{\lambda}_2^3\Big(\frac{\pi\beta^2}{36}\Big)dl\int\frac{dt}{a}\ \cos 2\beta\Phi_{s\Lambda'}(t)\,. \qquad (3.24)
\end{aligned}
$$

In order to restore the original cut-off to Λ we rescale the energy $\omega'=(\Lambda/\Lambda')\omega\simeq(1+dl)\omega$. In order to keep the product ωt intact, we have to rescale time in the opposite way, $t'=(1-dl)t$. The effective action is of the same form as the bare one but with a renormalized strength of coupling. Hence neglecting the $O(dl^2)$ terms,

$$
S_{I\ eff}[\Phi_\Lambda]\ =\ \tilde{\lambda}_0(1+dl)\int\frac{dt}{a}\ \mathbf{1}-\tilde{\lambda}_1\Big(1+\Big(1-\frac{\beta^2}{4\pi}\Big)dl\Big)\int\frac{dt}{a}\ \cos\beta\Phi_{s\Lambda'}(t)
$$

$$
\begin{aligned}
&-\tilde{\lambda}_2\Big(1+\Big(1-\frac{\beta^2}{\pi}\Big)dl\Big)\int\frac{dt}{a}\;\cos 2\beta\Phi_{s\Lambda'}(t)\\
&+\tilde{\lambda}_1^2\frac{\beta^2}{2\pi}dl\int\frac{dt}{a}\;\cos 2\beta\Phi_{s\Lambda'}(t)-\tilde{\lambda}_1^2\frac{\beta^2}{2\pi}dl\int\frac{dt}{a}\;\mathbf{1}\\
&+\tilde{\lambda}_2^2\frac{2\beta^2}{\pi}dl\int\frac{dt}{a}\;\cos 4\beta\Phi_{s\Lambda'}(t)-\tilde{\lambda}_2^2\frac{4\beta^2}{2\pi}dl\int\frac{dt}{a}\;\mathbf{1}\\
&+2\frac{\tilde{\lambda}_1\tilde{\lambda}_2}{\pi}\beta^2 dl\int\frac{dt}{a}\;\cos 3\beta\Phi_{s\Lambda'}(\tilde{R})-2\frac{\tilde{\lambda}_1\tilde{\lambda}_2}{\pi}\beta^2 dl\int\frac{dt}{a}\;\cos \beta\Phi_{s\Lambda'}(\tilde{R})\\
&+\tilde{\lambda}_1^3\Big(\frac{\pi\beta^2}{144}\Big)dl\int\frac{dt}{a}\;\cos 3\beta\Phi_{s\Lambda'}(t)+\tilde{\lambda}_1^3\Big(\frac{\pi\beta^2}{144}\Big)dl\int\frac{dt}{a}\;\cos \beta\Phi_{s\Lambda'}(t)\\
&+\tilde{\lambda}_2^3\Big(\frac{\pi\beta^2}{36}\Big)dl\int\frac{dt}{a}\;\cos 6\beta\Phi_{s\Lambda'}(t)+\tilde{\lambda}_2^3\Big(\frac{\pi\beta^2}{36}\Big)dl\int\frac{dt}{a}\;\cos 2\beta\Phi_{s\Lambda'}(t)\,.
\end{aligned}
\tag{3.25}
$$

Note that although initially we considered perturbation with tachyonic zero, first and second modes, the non-linear RG flow equations force the coefficients of various other boundary fields with higher harmonics to evolve. We see from above expression that the third and fourth harmonics evolve at second order and the third and sixth harmonics evolve at third order. Also notice that there is no mixing between λ_1 and λ_2 at third order in the coupling. The beta functions can be extracted from the above expression and are given by

$$
\begin{aligned}
\beta_0 &= \frac{d\tilde{\lambda}_0}{dl}=\tilde{\lambda}_0-\frac{\beta^2}{2\pi}\tilde{\lambda}_1^2-\frac{2\beta^2}{\pi}\tilde{\lambda}_2^2\,,\\
\beta_1 &= \frac{d\tilde{\lambda}_1}{dl}=-\Big(1-\frac{\beta^2}{4\pi}\Big)\tilde{\lambda}_1-\frac{2\beta^2}{\pi}\tilde{\lambda}_1\tilde{\lambda}_2+\frac{\pi\beta^2}{144}\tilde{\lambda}_1^3\,,\\
\beta_2 &= \frac{d\tilde{\lambda}_2}{dl}=-\Big(1-\frac{\beta^2}{\pi}\Big)\tilde{\lambda}_2+\frac{\beta^2}{2\pi}\tilde{\lambda}_1^2+\frac{\pi\beta^2}{36}\tilde{\lambda}_2^3\,.
\end{aligned}
\tag{3.26}
$$

Now we take $R_{25}=\sqrt{3}$, where the perturbation with λ_1 and λ_2 becomes least relevant and irrelevant respectively. Perturbation with the identity operator (associated with λ_0) in the RG picture just adds a constant to the action. Typically the coefficient of the least relevant operator grows at the fastest rate at the beginning, and drives the system towards the fixed point. The final shape of the soliton is determined by where this fixed point is. Rescaling of $\beta:\beta\to\sqrt{4\pi}\beta$ in (4.1) results in

$$
\begin{aligned}
\beta_0 &= \frac{d\tilde{\lambda}_0}{dl}=\tilde{\lambda}_0-\frac{2}{3}\tilde{\lambda}_1^2-\frac{8}{3}\tilde{\lambda}_2^2\,,\\
\beta_1 &= \frac{d\tilde{\lambda}_1}{dl}=-\frac{2}{3}\tilde{\lambda}_1-\frac{8}{3}\tilde{\lambda}_1\tilde{\lambda}_2+\frac{\pi^2}{108}\tilde{\lambda}_1^3\,,\\
\beta_2 &= \frac{d\tilde{\lambda}_2}{dl}=\frac{1}{3}\tilde{\lambda}_2+\frac{2}{3}\tilde{\lambda}_1^2+\frac{\pi^2}{27}\tilde{\lambda}_2^3\,.
\end{aligned}
\tag{3.27}
$$

Typically single a Sine-Gordon boundary perturbation doesn't lead to a nearby fixed point and the theory typically flows to infinity. Typically λ flows all the way from 0 to ∞ under renormalization. The boundary conditions look like Neumann at very high energy (UV) but like Dirichlet at low energy (IR). The field Φ satisfy Neumann boundary conditions close to the boundary, but feels Dirichlet boundary conditions instead far from it. But as in the next section we will see that Sine-Gordon model with two frequencies we considered above has a nearby fixed point.

The situation is very analogous to string field theory. Suppose we are trying to construct the lump solution on a circle of radius R_{25}. In this case we need to start with an initial tachyon field for which the coefficient of $\cos(\Phi/R_{25})$ is non-zero, and then use the equations of motion of string field theory iteratively to improve the solution. This generates all higher harmonics as well as the tachyon zero momentum mode as we saw in our RG analysis as well. But if from the beginning we introduce the tachyon zero momentum mode in the initial configuration, then the iterative process drives the solution towards the vacuum solution where the tachyon is constant instead of the one lump solution. In the RG picture perturbations with the identity operator are generally regarded as uninteresting, as they just add a constant to the action. Also, in the peturbative definition via correlation functions, the zero mode has no effect.

4 The lump profile and boundary entropy

In this section we will calculate the boundary entropy perturbatively in leading order and compare the result with its exact value. We follow the method of [29]. Let us recall the RG equations (3.26) obtained in the last section (up to second order in coupling constants)

$$\begin{aligned}
\beta_0 &= \frac{d\tilde{\lambda}_0}{dl} = \tilde{\lambda}_0 - 2(1-y_1)\tilde{\lambda}_1^2 - 8(1-y_1)\tilde{\lambda}_2^2 = 0\,, \\
\beta_1 &= \frac{d\tilde{\lambda}_1}{dl} = -y_1\tilde{\lambda}_1 - 8(1-y_1)\tilde{\lambda}_1\tilde{\lambda}_2 = 0\,, \\
\beta_2 &= \frac{d\tilde{\lambda}_2}{dl} = -y_2\tilde{\lambda}_2 + 2(1-y_1)\tilde{\lambda}_1^2 = 0\,,
\end{aligned} \tag{4.1}$$

where $\Delta_i = 1 - y_i$ is the scaling dimension of the corresponding harmonic. The above RG equations satisfies right ϵ-expansion behaviour according to [38]. The first step will be to solve these equations for the bare couplings, $\tilde{\lambda}_0$, $\tilde{\lambda}_1$, and $\tilde{\lambda}_2$, as functions of renormalized couplings at the scale set by R, $\tilde{\lambda}_0(R)$, $\tilde{\lambda}_1(R)$, and $\tilde{\lambda}_2(R)$.

The Pfaffian differential equation obtained from the above set of equations is

$$[y_1\tilde{\lambda}_1 + 8(1-y_1)\tilde{\lambda}_1\tilde{\lambda}_2](d\tilde{\lambda}_0 + d\tilde{\lambda}_2) + [\tilde{\lambda}_0 - 8(1-y_1)\tilde{\lambda}_2^2 - (4y_1-3)\tilde{\lambda}_2]d\tilde{\lambda}_1 = 0\,, \tag{4.2}$$

which is needed in order to satisfy the integrability condition given by

$$
\begin{aligned}
&[y_1\tilde{\lambda}_1 + 8(1-y_1)\tilde{\lambda}_1\tilde{\lambda}_2][24(1-y_1)\tilde{\lambda}_2 + 5y_1 - 3] + 8(1-y_1)\tilde{\lambda}_1[\tilde{\lambda}_0 - 8(1-y_1)\tilde{\lambda}_2^2 - (4y_1-3)\tilde{\lambda}_2] \\
&+[y_1\tilde{\lambda}_1 + 8(1-y_1)\tilde{\lambda}_1\tilde{\lambda}_2][1 - y_1 - 8(1-y_1)\tilde{\lambda}_2] = 0\,.
\end{aligned} \tag{4.3}
$$

For $\tilde{\lambda}_1 \neq 0$, we have the following relation between $\tilde{\lambda}_0$ and $\tilde{\lambda}_2$

$$
64(1-y_1)^2\tilde{\lambda}_2^2 + 8(1-y_1)(1+2y_1)\tilde{\lambda}_2 + 8(1-y_1)\tilde{\lambda}_0 - 2y_1(1-2y_1) = 0\,. \tag{4.4}
$$

The branch containing the real solution of $\tilde{\lambda}_2$ satisfies the following,

$$
8(1-y_1)\tilde{\lambda}_0 + 2y_1(2y_1-1) \leq (2y_1+1)^2/4\,, \tag{4.5}
$$

which implies $\tilde{\lambda}_0 \leq 0.34$ for $R_{25} = \sqrt{3}$, concistent with our results for the desired IR fixed point discussed below. Combining β_0 and β_2 in (4.1) into

$$
(d\tilde{\lambda}_0 + d\tilde{\lambda}_2)/[\tilde{\lambda}_0 - 8(1-y_1)\tilde{\lambda}_2^2 - (4y_1-3)\tilde{\lambda}_2] = dl\,, \tag{4.6}
$$

and using the relation (4.4), we get

$$
8(1-y_1)d\tilde{\lambda}_2/[8(1-y_1)\tilde{\lambda}_2 + 2y_1 - 1] = dl\,, \tag{4.7}
$$

for $\tilde{\lambda}_2 \neq \tilde{\lambda}_2^*$. Integrating the above expression from a to R, we obtain the following expression for the bare coupling $\tilde{\lambda}_2$

$$
\left[\tilde{\lambda}_2\Big(\frac{R}{a}\Big)^{y_2}\right] = -\tilde{\lambda}_2^*\left[\Big(\frac{2y_1-1}{y_1} - \frac{\tilde{\lambda}_2(R)}{\tilde{\lambda}_2^*}\Big)\Big(\frac{R}{a}\Big)^{y_2-1} - \Big(\frac{2y_1-1}{y_1}\Big)\Big(\frac{R}{a}\Big)^{y_2}\right]\,, \tag{4.8}
$$

where $\tilde{\lambda}_2^* = -y_1/8(1-y_1)$ and $1 - y_2 = \Delta_2 = 4y_1 - 3$ is the scaling dimension of the second tachyon harmonic. For $R_{25} = \sqrt{3}$, the perturbation with the second harmonic is least irrelevant and is added to improve the shape of the profile of the lump at IR fixed point. Being irrelavant the perturbing operator with $\tilde{\lambda}_2$ decays out. This can be seen by considering the perturbation expansion of the two-point correlation function of bosonic exponents. The integral appearing in the first nonvanishing correction to this correlation function converges at large distances for an irrelevant operator. This implies that the perturbation expansion does not contain IR singularities; so if the bare coupling constant is small, its effect will remain small and will not be amplified in the process of renormalization. In the limit the renormalization scale, $R \to \infty$,

$$
\tilde{\lambda}_2 = \frac{a}{R}\tilde{\lambda}_2(R) - \Big(\frac{a}{R} - 1\Big)\Big(\frac{2y_1-1}{y_1}\Big)\tilde{\lambda}_2^* \to -0.125\,, \tag{4.9}
$$

which is very close to the result of [11] for the second harmonic at the stable minimum of the tachyon potential. On the other hand, the other two operators which are relevant for $R_{25} = \sqrt{3}$ evolve to the values: $\tilde{\lambda}_0^* = 0.25$, $\tilde{\lambda}_1^* \approx -0.35$, which are also very close to the results obtained by the level truncation technique [11]. Hence the lump profile is given by

$$T(x_{25}) = 0.25 - 0.35\cos(\frac{1}{R_{25}}x_{25}) - 0.125\cos(\frac{2}{R_{25}}x_{25}) \tag{4.10}$$

as a function of x_{25}. The profile is shown in figure 1. The profile has finite size in contrast to the result of [27].

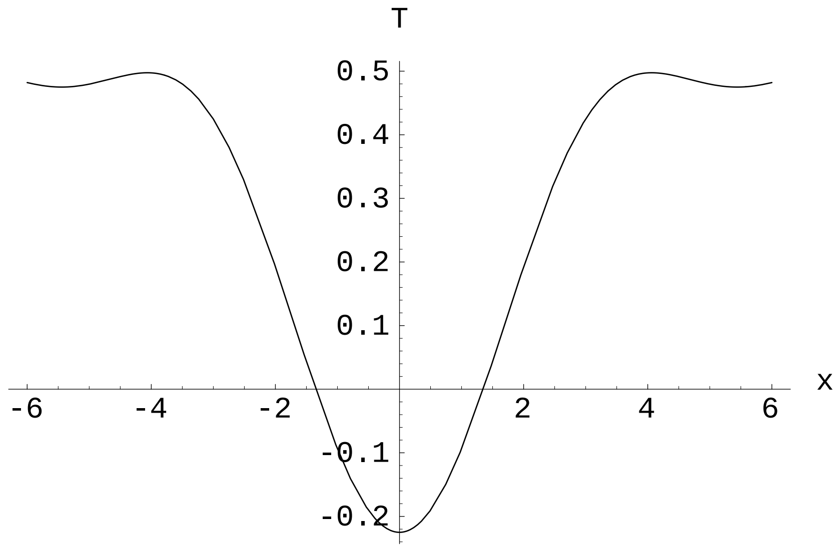

Figure 1: The lump solution with $\lambda_0, \lambda_1, \lambda_2$ switched on at $R_{25} = \sqrt{3}$.

Before returning to the calculation of boundary entropy, notice that in the RG equations (4.1) in the limit $R \to \infty$, the linear terms in β_1 and β_2 agree. The reason for these to disagree with β_0 is that we chose a positive sign in front of λ_0 in the beginning. The RG equations up to linear order are $\frac{d\lambda_i}{dl} = \pm\Delta_i\lambda_i + O(\lambda_i^2)$, where the sign in front of the universal linear term depends on the sign of perturbation. RG analysis with the identity operator is trivial and it does not mix with other operators. Also as we will discuss soon its contribution goes to ground state energy correction only, not to boundary entropy.

Next we calculate the bare $\tilde{\lambda}_0$ and $\tilde{\lambda}_1$ in terms of the renormalized couplings. In order to obtain the expression for bare coupling, $\tilde{\lambda}_0$ in terms of the renormalized couplings we use (4.4) setting the scale at UV cutoff:

$$\tilde{\lambda}_0(a) = -\Big[8(1-y_1)\tilde{\lambda}_2^2(a) + (1+2y_1)\tilde{\lambda}_2(a) + \frac{2y_1(2y_1-1)}{8(1-y_1)}\Big]\,. \tag{4.11}$$

Inserting (4.8) for the bare coupling, $\tilde{\lambda}_2$, we get the following expression for $\tilde{\lambda}_0$

$$\Big[\Big(\frac{R}{a}\Big)\tilde\lambda_0\Big] = -\frac{2y_1-1}{8(1-y_1)}\Big[2y_1 - y_1\frac{\tilde\lambda_2(R)}{\tilde\lambda_2^*} + \frac{a}{R}\Big(1-\frac{2y_1}{2y_1-1}\frac{\tilde\lambda_2(R)}{\tilde\lambda_2^*}\Big)\Big]. \tag{4.12}$$

Inserting the above expression for bare coupling, $\tilde\lambda_0$ into the Pfaffian equation (4.2) we get

$$\frac{[y_1+8(1-y_1)\tilde\lambda_2]^2 d\tilde\lambda_2}{8(1-y_1)\tilde\lambda_2^2 + (3y_1-1)\tilde\lambda_2 + y_1(2y_1-1)/[8(1-y_1)]} = -\frac{d\tilde\lambda_1}{\tilde\lambda_1}, \tag{4.13}$$

which, assuming $\tilde\lambda_2 \neq \tilde\lambda_2^*$, simplifies to

$$\Big(\frac{8(1-y_1)+y_1}{8(1-y_1)+2y_1-1}\Big) d[8(1-y_1)\tilde\lambda_2] = -\frac{d\tilde\lambda_1}{\tilde\lambda_1}. \tag{4.14}$$

Integrating both sides we obtain

$$8(1-y_1)\tilde\lambda_2 + (1-y_1)\ ln[8(1-y_1)\tilde\lambda_2 + 2y_1 - 1] = -ln\ \tilde\lambda_1 + ln\ c, \tag{4.15}$$

where the integration constant determines the correct trajectory that passes through UV and desired IR fixed points. One might wonder since many different theories flow to the same fixed point, that we cannot write the bare couplings in terms of renormalized couplings. In fact, here as we are starting with the correct set of RG equations which inherently have the right UV fixed point as a trivial solution So over an infinitesimal segment from the UV fixed point we are automatically on the right trajectory. What we need to worry about is that as we integrate those equations we don't deviate from the right trajectory and miss the IR fixed point. We reach the desired fixed point by determining the correct value of the integration constant. To determine the constant that allows us to reach the desired *nearby* fixed point as the end point of the flow we substitute the couplings with their fixed point values

$$\tilde\lambda_1^* = -\frac{\sqrt{(3-4y_1)y_1}}{4(1-y_1)}, \qquad \tilde\lambda_2^* = -\frac{y_1}{8(1-y_1)}, \tag{4.16}$$

into (4.15) resulting in the following expression for the bare coupling, $\tilde\lambda_1$

$$\begin{aligned}\Big[\tilde\lambda_1\Big(\frac{R}{a}\Big)^{y_1}\Big] &= e^{-(1-y_1)}\Big(\frac{1-y_1}{1-2y_1}\Big)^{1-y_1}\frac{\sqrt{y_1(3-4y_1)}}{4(1-y_1)}\Big[\Big(\frac{R}{a}\Big) + \Big(\frac{R}{a}\Big)\frac{y_1(1-y_1)}{2y_1-1}\frac{\tilde\lambda_2(R)}{\tilde\lambda_2^*} \\ &\quad -(2y_1-1)\Big(1-\frac{y_1}{2y_1-1}\frac{\tilde\lambda_2(R)}{\tilde\lambda_2^*}\Big) - y_1(1-y_1)\frac{\tilde\lambda_2(R)}{\tilde\lambda_2^*}\Big],\end{aligned} \tag{4.17}$$

where we have used (4.8).

Having solved the renormalization group equations for the bare couplings as functions of renormalized couplings, given by (4.12), (4.17), and (4.8) we turn to the calculation for boundary entropy which measures the tension of the corresponding D-brane and test the *g-conjecture*, which in turn is related to the minimisation of the action in the space of open string fields. The boundary has its own free energy proportional to the size of the system which diverges linearly with scale $l = \frac{R}{a}$ in thermodynamic limit. The boundary free energy gets another contribution independent of scale which counts the boundary degrees of freedom, measured by $ln\ g$,

$$F_B L = -ln\ Z = f_B L - ln\ g\,. \tag{4.18}$$

We first perturbatively expand the partition function Z to $O(\tilde{\lambda}^3)$. We arrive at a UV expansion of the form which schematically looks like

$$ln\ Z = \sum_n c_n(\tilde{\lambda}_i R^{y_i})^n = f_B\Big[\tilde{\lambda}_i\Big(\frac{R}{a}\Big)^{y_i}\Big]^{\frac{1}{y_i}} + ln\ g + \Big[\tilde{\lambda}_i\Big(\frac{R}{a}\Big)^{y_i}\Big]^{-ve}, \tag{4.19}$$

where $\Delta_i = 1 - y_i$ is the scaling dimension. We discard terms linear in negative powers of scale and nonuniversal UV divergent terms which are linear in scale as they correspond to ground state energy corrections. The remaining terms have weak dependence on the scale set by R, which acts as an IR cut-off, as we can absorb them approximately in terms of the renormalized coupling $\tilde{\lambda}_i(R)$ using the relations (4.12), (4.17), and (4.8) obtained by solving simultaneous β-function equations. Thus in the $R \to \infty$ limit (when $\tilde{\lambda}_i(R) \to \tilde{\lambda}_i^*$), these terms give the contribution to the boundary entropy. Note that the weak IR cut-off dependence is consistent with the renormalization group.

To cubic order in coupling the partition function is given by

$$\begin{aligned}\frac{Z}{Z_0} \ &\approx\ 1 + \frac{\tilde{\lambda}_i\tilde{\lambda}_j}{2!}\int dt_1 dt_2 \mathcal{T}\langle\cos\beta_i\Phi(t_1)\cos\beta_j\Phi(t_2)\rangle \\ &\quad + \frac{\tilde{\lambda}_i\tilde{\lambda}_j\tilde{\lambda}_k}{3!}\int dt_1 dt_2 dt_3 \mathcal{T}\langle\cos\beta_i\Phi(t_1)\cos\beta_j\Phi(t_2)\cos\beta_k\Phi(t_3)\rangle\,.\end{aligned} \tag{4.20}$$

Since $\langle\cos\beta_i\Phi(t_1)\rangle \propto \left(\frac{R}{a}\right)^{-\Delta_i}$, this vanishes in the thermodynamic limit, $R \to \infty$. Here Z_0 is the partition function at UV fixed point. Any n-point correlation function of Sine-Gordon perturbation, the electro-nonneutral ($\sum_n \beta_n \neq 0$ in (3.12)) part has an overall factor of $\left(\frac{R}{a}\right)^{-\Delta_i}$ that cannot be compensated using definition of renormalized coupling $\tilde{\lambda}_i(R)$; therefore in above perturbative expansion contributions come from electro-neutral parts only such as $O(\tilde{\lambda}_1^2)$, $O(\tilde{\lambda}_2^2)$, and $O(\tilde{\lambda}_1^2\tilde{\lambda}_2)$. Clearly there is no contribution from the identity operator of any order as its contribution can be considered to be going to ground state energy correction.

The perturbing operator has two point function given by

$$\langle\cos\beta_i\Phi(t_1)\cos\beta_i\Phi(t_2)\rangle = \frac{1}{2}\Big(\frac{a}{t_1 - t_2}\Big)^{2(1-y_i)}\,. \tag{4.21}$$

So the quadratic contribution to Z/Z_0 turns out to be (considering half-cylinder geometry)

$$\begin{aligned} Z_2 &= \frac{1}{4}\sum_{i=1,2} a^{-2y_i}\tilde{\lambda}_i^2 \int \frac{d\tau_1 d\tau_2}{|\frac{R}{\pi}\sin\frac{\pi}{R}(\tau_1-\tau_2)|^{2(1-y_i)}} \\ &= \frac{1}{4}\sum_{i=1,2} a^{-2y_i}\tilde{\lambda}_i^2 R\int_{-R/2}^{R/2} \frac{d\tau_1}{|\frac{R}{\pi}\sin\frac{\pi}{R}(\tau)|^{2(1-y_i)}}\,, \end{aligned} \tag{4.22}$$

where $\tau = \tau_1 - \tau_2$ is the relative coordinate and $i = 1, 2$. The above integration is regularized by the cut-off $|\tau| > a$. Mapping the circle to infinite line: $u = \tan\frac{\pi}{R}\tau$ the above integration becomes

$$Z_2 = \frac{\pi}{4}\sum_{i=1,2}\tilde{\lambda}_i^2\Big(\frac{R}{\pi a}\Big)^{2y_i}\int_{|u|<\pi a/R}\frac{du}{(1+u^2)^{y_i}|u|^{2(1-y_i)}}\,. \tag{4.23}$$

Integration by parts results in

$$\begin{aligned} Z_2 &= \frac{\pi}{4}\sum_{i=1,2}\tilde{\lambda}_i^2\left(\frac{R}{\pi a}\right)^{2y_i} 2\left[-\frac{1}{(1+u^2)^{y_i}(1-2y_i)u^{1-2y_i}}\Bigg|_{\pi a/R}^{\infty} - \frac{2y_i}{1-2y_i}\int_0^\infty du\,\frac{u^{2y_i}}{(1+u^2)^{1+y_i}}\right] \\ &= \frac{\pi}{4}\sum_{i=1,2}\tilde{\lambda}_i^2\left(\frac{R}{\pi a}\right)^{2y_i} 2\left[\left(\frac{R}{\pi a}\right)^{1-2y_i} - \frac{2y_i}{1-2y_i}\,\frac{1}{2}B(y_i+\frac{1}{2},\frac{1}{2})\right], \end{aligned} \tag{4.24}$$

where $B(x, y)$ is the Euler beta function. The first term is linear in R and does not contribute to boundary entropy.

We will formally perform the computation as an expansion in y_1 supposing y_1 to be small and at the end we will set $y_1 = \frac{2}{3}$ corresponding to $R_{25} = \sqrt{3}$. It is not *a priori* obvious that such y_1 expansion will converge for finite value of y_1. However, as we will see, it does quite well in our case giving nice agreement with exact value of the ratio g_{IR}/g_{UV}. The situation is similar to computation of critical exponents in three dimensions where ϵ-expansion for $4-\epsilon$ dimensions converges for $\epsilon = 1$ and gives nice agreement with the experimental data. We note that we can reach nearby fixed point at the end of the RG flow with $y_1 = \frac{2}{3}$ indicating that y_1 is small enough.

Now in the small y_1 limit Z_2 becomes

$$Z_2 \approx -\frac{\pi^2 y_1}{2}\left[\tilde{\lambda}_1\Big(\frac{R}{\pi a}\Big)^{y_1}\right]^2, \tag{4.25}$$

where we have dropped the $O(\tilde{\lambda}^2)$ part which vanishes in the $R \to \infty$ limit as it has an overall factor of $\left(\frac{R}{a}\right)^{2y_2}$ with y_2 negative as can be seen by substituting (4.8) in (4.24).

The cubic part is given by

$$Z_3 = \frac{1}{3!}\frac{\tilde{\lambda}_1^2\tilde{\lambda}_2}{4}\int dt_1 dt_2 dt_3 \langle e^{i\beta\Phi(t_1)} e^{i\beta\Phi(t_2)} e^{-2i\beta\Phi(t_3)}\rangle \tag{4.26}$$

which on half-cylinder turns out to be

$$Z_3 = \frac{1}{3!}\frac{\tilde{\lambda}_1^2\tilde{\lambda}_2}{4}\; a^{-2y_1}a^{-y_2}\int d\tau_1 d\tau_2 d\tau_3 \; \frac{\left|\frac{R}{\pi}\sin\frac{\pi}{R}(\tau_1-\tau_2)\right|^{2(1-y_1)}}{\left|\left(\frac{R}{a}\right)^2 \sin\frac{\pi}{R}(\tau_2-\tau_3)\;\sin\frac{\pi}{R}(\tau_3-\tau_1)\right|^{1-y_2}} \,. \tag{4.27}$$

Now introducing the relative coordinates $\tilde{\tau}_1 = \tau_2 - \tau_3$ and $\tilde{\tau}_2 = \tau_1 - \tau_3$ and center-of-mass coordinate $\tilde{\tau} = (\tau_1+\tau_2+\tau_3)/3$ the above integral simplifies to

$$\begin{aligned} Z_3 \;&=\; \frac{1}{3!}\frac{\tilde{\lambda}_1^2\tilde{\lambda}_2}{4}\; a^{-2y_1}a^{-y_2} R\int_{|u_i|>\pi a/R} du_1 du_2 \left(\frac{R}{\pi}\right)^2 \frac{1}{(1+u_1^2)(1+u_2^2)} \\ &\cdot \left(\frac{R}{\pi}\right)^{2(1-y_1)}\left(\frac{R}{\pi}\right)^{-2(1-y_2)} \frac{1}{|u_1/\sqrt{1+u_1^2}|^{1-y_2}|u_2/\sqrt{1+u_2^2}|^{1-y_2}} \\ &\cdot \left|\left(\frac{u_1-u_2}{1+u_1u_2}\right)\Big/\sqrt{1+\left(\frac{u_1-u_2}{1+u_1u_2}\right)^2}\right|^{2(1-y_1)} , \end{aligned} \tag{4.28}$$

where $u_i = \tan\frac{\pi}{R}\tilde{\tau}_i$ with a regularization $|u_i| > \pi a/R$, $|u_1 - u_2| > \pi a/R$. The above expression can be written in a more convenient form given by

$$Z_3 = \frac{\pi}{4\times 3!}\left[\tilde{\lambda}_1\left(\frac{R}{\pi a}\right)^{y_1}\right]^2\left[\tilde{\lambda}_2\left(\frac{R}{\pi a}\right)^{y_2}\right]\int_{|u_i|>\pi a/R} du_1 du_2 \; \frac{|u_1-u_2|^{2(1-y_1)}}{[(1+u_1^2)(1+u_2^2)]^{y_1}(|u_1||u_2|)^{1-y_2}} \,. \tag{4.29}$$

Let us first concentrate on terms of $O[(R/a)^{3y_1}]$ and the ground state energy contribution from terms of $O(R/a)$. To achieve that we make a change of variables: $u_2 = vu_1$. The above integral becomes

$$Z_3 = \frac{\pi}{4\times 3!}\left[\tilde{\lambda}_1\left(\frac{R}{\pi a}\right)^{y_1}\right]^2\left[\tilde{\lambda}_2\left(\frac{R}{\pi a}\right)^{y_2}\right]\int_{|u_1|>u_0(v)} du_1 dv \; \frac{(1-v)^{2(1-y_1)}}{|v|^{4(1-y_1)}|u_1|^{5-6y_1}[(1+u_1^2)(1+u_1^2v^2)]^{y_1}} \,. \tag{4.30}$$

where $u_0(v) \equiv \max\{\frac{\pi a}{R}, \frac{\pi a}{R|v|}, \frac{\pi a}{R|1-v|}\}$ is the new cut-off. Integration by parts with respect to u_1 gives

$$Z_3 \;=\; \frac{\pi}{4\times 3!}\left[\tilde{\lambda}_1\left(\frac{R}{\pi a}\right)^{y_1}\right]^2\left[\tilde{\lambda}_2\left(\frac{R}{\pi a}\right)^{y_2}\right]\int_{-\infty}^{\infty} dv$$

$$
\cdot\frac{|1-v|^{2(1-y_1)}}{|v|^{4(1-y_1)}}\left[\frac{-2}{[(1+u_1^2)(1+v^2u_1^2)]^{y_1}(4-6y_1)u_1^{4-6y_1}}\Bigg|_{u_0(v)}^{\infty}\right.
$$
$$
\left.-\frac{2y_1}{4-6y_1}\int_{-\infty}^{\infty}\frac{du_1}{[(1+u_1^2)(1+v^2u_1^2)]^{y_1}|u_1|^{5-6y_1}}\left(\frac{u_1^2}{1+u_1^2}+\frac{v^2u_1^2}{1+v^2u_1^2}\right)\right]. \tag{4.31}
$$

From the surface term we will get contribution of $O\left[\left(\frac{R}{\pi a}\right)\right]$ which has no effect on boundary entropy. The rest is UV finite and we can remove the cut-off. However there will be a contribution of $O\left[\left(\frac{R}{a}\right)^{2y_1}\right]$ coming from $O\left[\left(\frac{a}{R}\right)^{y_1}\right]$ term in the denominator. To extract this contribution we perform the following calculation.

We first introduce step functions into the integral (4.29)

$$
I(\epsilon)=\int_{-\infty}^{\infty}du_1du_2\,\frac{|u_1-u_2|^{2(1-y_1)}}{[(1+u_1^2)(1+u_2^2)]^{y_1}(|u_1||u_2|)^{4(1-y_1)}}\theta(u_1^2-\epsilon^2)\theta(u_2^2-\epsilon^2)\theta[(u_1-u_2)^2-\epsilon^2]\,, \tag{4.32}
$$

where $\epsilon\equiv a\pi/R$ is the cut-off. It is convenient to differentiate the integral with respect to ϵ:

$$
\frac{dI(\epsilon)}{d\epsilon}\approx\frac{6}{|\epsilon|^{4(1-y_1)}}\int_{-\infty}^{\infty}\frac{du_1}{(1+u_1^2)^{y_i}|u_1|^{2(1-y_i)}}\;\theta(u_1^2-\epsilon^2)\,. \tag{4.33}
$$

The integral is identical to (4.23) with the same cut-off. So using previous result obtained for Z_2 we get

$$
\frac{dI(\epsilon)}{d\epsilon}=\frac{6}{\epsilon^{4(1-y_1)}}\times 2[\epsilon^{-(1-2y_1)}-\pi y_1]\,, \tag{4.34}
$$

that gives

$$
I(\epsilon)=\frac{12}{6y_1-4}\left(\frac{R}{\pi a}\right)^{4-6y_1}-\frac{12\pi y_1}{y_2}\left(\frac{R}{\pi a}\right)^{-y_2}+\text{const.} \tag{4.35}
$$

The second part gives the desired $O\left[\left(\frac{R}{a}\right)^{2y_1}\right]$ behaviour that has weak cut-off dependence. But the first part is of no interest to us since it is the ground state energy contribution from Z_3. The constant part comes from evaluating the integral (4.31) throwing away the surface term. Writing the integral in terms of old variables u_1,u_2 we get

$$
-\frac{2\pi y_1}{4\times 3!(4-6y_1)}\left[\tilde{\lambda}_1\left(\frac{R}{\pi a}\right)^{y_1}\right]^2\left[\tilde{\lambda}_2\left(\frac{R}{\pi a}\right)^{y_2}\right]
$$
$$
\cdot\int_{-\infty}^{\infty}\frac{du_1du_2\;|u_1-u_2|^{2(1-y_1)}}{[(1+u_1^2)(1+u_2^2)]^{y_1}|u_1u_2|^{4(1-y_1)}}\left(\frac{u_1^2}{1+u_1^2}+\frac{u_2^2}{1+u_2^2}\right)
$$

$$= -\frac{\pi y_1}{3!(4-6y_1)}\left[\tilde{\lambda}_1\left(\frac{R}{\pi a}\right)^{y_1}\right]^2\left[\tilde{\lambda}_2\left(\frac{R}{\pi a}\right)^{y_2}\right]\int_{-\infty}^{\infty}\frac{du_1 du_2\ |u_1-u_2|^{2(1-y_1)}}{[(1+u_1^2)(1+u_2^2)]^{y_1}|u_1u_2|^{4(1-y_1)}}\left(\frac{u_1^2}{1+u_1^2}\right). \tag{4.36}$$

We notice that in small y_1 limit, while performing the u_2 integration, the y_1 factor in front is cancelled by a divergence from $u_2 \approx 0$:

$$2\int_{\pi a/R}^{\infty}\frac{du_2\ (u_1-u_2)^{2(1-y_1)}}{(1+u_2^2)^{y_1}|u_2|^{4(1-y_1)}} \approx -\frac{2|u_1|^{2(1-y_1)}}{4y_1-3}\left(\frac{\pi a}{R}\right)^{4y_1-3}. \tag{4.37}$$

Hence in small y_1 limit, the constant part of (4.35) becomes

$$-\frac{2\pi y_1}{3!(4-6y_1)(3-4y_1)}\left[\tilde{\lambda}_1\left(\frac{R}{\pi a}\right)^{y_1}\right]^2\left[\tilde{\lambda}_2\left(\frac{R}{\pi a}\right)^{y_2}\right]\left(\frac{\pi a}{R}\right)^{4y_1-3}\int_{-\infty}^{\infty}\frac{du_1}{(1+u_1^2)^{y_1}|u_1|^{2(1-y_1)}}\left(\frac{u_1^2}{1+u_1^2}\right)$$
$$\approx -\frac{\pi^2 y_1}{3!\times 6}\left[\tilde{\lambda}_1\left(\frac{R}{\pi a}\right)^{y_1}\right]^2\left[\tilde{\lambda}_2\left(\frac{R}{\pi a}\right)^{y_2}\right]\left(\frac{R}{\pi a}\right)^{-y_2}. \tag{4.38}$$

So ignoring higher orders in y_1 the final expression for Z/Z_0 contributing to boundary entropy is given by

$$\frac{Z}{Z_0} = 1-\frac{\pi^2}{2}y_1\left[\tilde{\lambda}_1\left(\frac{R}{\pi a}\right)^{y_1}\right]^2-\frac{\pi^2 y_1}{3!}\left[\frac{1}{6}-\left(1+\frac{4y_1}{3}\right)\right]\left[\tilde{\lambda}_1\left(\frac{R}{\pi a}\right)^{y_1}\right]^2\left[\tilde{\lambda}_2\left(\frac{R}{\pi a}\right)^{y_2}\right]\left(\frac{R}{\pi a}\right)^{-y_2}. \tag{4.39}$$

Having already obtained the expression for the bare couplings in terms of the renormalized couplings as given by (4.8) and (4.17) we can rewrite the above result in terms of renormalized couplings only. As $R \to \infty$, $\tilde{\lambda}_i(R) \to \tilde{\lambda}_i^*$. Hence the desired ratio between ground state entropy at IR and UV fixed points turns out to be (in the leading order in y_1)

$$r_p \equiv \frac{g_{\rm IR}}{g_{\rm UV}} = \lim_{R\to\infty}\frac{Z}{Z_0} \approx 1 - 0.50088964\ y_1^2\,. \tag{4.40}$$

Notice that the change in g is negative, implying decrease of g under flow between UV to IR point. Also the ratio becomes unity for exactly marginal case, *i.e.* where $y_1 = 0$, giving a line of fixed points. On the other hand the exact result is given by [32]

$$r_e \equiv \frac{g_{\rm IR}}{g_{\rm UV}} = \sqrt{1-y_1}\,. \tag{4.41}$$

For $R_{25} = \sqrt{3}$, perturbative result is $r_p \approx 0.65695$ compared to the exact result, $r_e \approx 0.57735$; so our result is within 13%. On the other hand for $R_{25} = 1.1$, *i.e.* when y_1 is very small and the perturbation by the first tachyon harmonic is nearly marginal, the perturbative result is $r_p \approx 0.98491$ compared to the exact result, $r_e \approx 0.90909$; the perturbative result is now obtained to an accuracy of 8% of the exact result.

5 Multicriticality, $U(\infty)$ and Kondo picture of IR fixed point

In the effective Landau-Ginzburg description of two dimensional conformal field theory,

$$\mathcal{L} = \int d^2x \, (\frac{1}{2}(\partial\Phi)^2 + V(\Phi)) \, , \tag{5.42}$$

Φ being the order parameter for some physical system, the extrema of the general polynomial interaction correspond to various critical phases of the system. Many systems possess a $\Phi \to -\Phi$ symmetry with an even polynomial interaction $V(\Phi) = \sum_m g_m \Phi^{2m}$. For a polynomial $V(\Phi)$ of degree $2(m-1)$, this ensures existence of $(m-1)$ minima separated by $(m-2)$ maxima. Several critical phases can coexist if the corresponding extrema coincide. Hence the most critical potential is a monomial in Φ and the $(m-1)$ critical behaviour of the theory is given by the interaction:

$$\mathcal{L}_{int} = g \int d^2x \; \Phi^{2(m-1)} \, . \tag{5.43}$$

By comparing the structure of the operator algebra of the above bulk critical theory with that of the unitary diagonal minimal model $M_{m+1,m}$, characterized by central charge $c(m) = 1 - \frac{6}{m(m+1)}$ for $m = 3, 4, \ldots$, Zamolodchikov [39] has shown that each $(m-1)$ multicritical behaviour of the theory (5.42) is nothing but a minimal model, $M_{m+1,m}$.

Generalizing this concept to boundary conformal field theory we consider the boundary potential of the form

$$V(\Phi) = \sum_k g_k \Phi^{2k}(0) \, . \tag{5.44}$$

The bulk theory is still the $c = 1$ (*i.e.* $M_{\infty+1,\infty}$) minimal model throughout the flow. Near the UV fixed point, as we turn on the boundary interaction (which can be considered as even polynomial interaction of order infinity and hence an indication of underlying infinite number of critical phases of the system) the system flows to the IR near which the system has finite multicritical behaviour. The strength of multicriticality (*i.e.* how many critical phases can coexist) depends on number of couplings involved, *i.e.* on dimensionality of the space of coupling constants. In other words, expanding the Sine-Gordon polynomial about $X_{25}(0,\tau) = 0$ near the IR fixed point, the potential has the following form:

$$V(\Phi) = (\sum_n \tilde{\lambda}_n^*) - (\sum_n \tilde{\lambda}_n^* \frac{n^2}{2!R^2})\Phi(0)^2 + (\sum_n \tilde{\lambda}_n^* \frac{n^4}{4!R^4})\Phi(0)^4 - \ldots \equiv (-1)^k \sum_k g_k^* \Phi(0)^{2k} \, . \tag{5.45}$$

Now using the results from previous section, $g_0^* = \sum_n (-1)^n \tilde{\lambda}_n^* \approx 0.25 - 0.3535 + 0.25 - \ldots = 0.146 - \ldots \sim 0$, so one can consider $g_0^* \to 0$ for a large number of couplings turned on. For the perturbation with the first three tachyon harmonics (3.8), *i.e.* $n \leq 2$, we observe that for $R_{25} > 1$, only $g_1^* \sim 0.108 \neq 0$, $g_2^* \sim 0.051 \neq 0$, and all other $g_i^* \sim 0.0008 \sim 0$ indicating $V(\Phi) \sim g_0^* + g_1^* \Phi^2 + g_2^* \Phi^4$ and hence the existence of two critical phases of the system at the IR fixed point. Also for R_{25} close to one (*i.e.* the self-dual radius), $g_1^* \sim 0.325$, $g_2^* \sim 0.152 \neq 0$ showing the two phases again. As we turn on more and more couplings, we will see that lower order coefficients of $V(\Phi)$ become zero, but some higher order coefficient becomes nonzero producing a shift in the degree of the polynomial effective interaction indicating the presence of more and more critical phases. For $n \to \infty$ the effective interaction is basically a monomial of very large order. We can expect that by turning on all possible couplings (*i.e.* infinite number of tachyon modes) we can probe the ∞-multicritical behaviour of the system.

Note that the values of coupling constants g_k^* are meaningful only in the context of a particular renormalization scheme and when the composite fields Φ^{2k} are defined. However the multicritical behaviour observed is RG scheme independent [39]. For the effective polynomial interaction $V(\Phi) = \sum_{k=1}^N g_k \Phi^{2k}(0)$, the m-critical behaviour (*i.e.* presence of m degenerate minima) of the potential is confined to hypersurfaces S_m, $m = 2, 3, 4, \ldots, N$ of codimension $m - 2$, *i.e.* of dimension $N + 2 - m$. The $N + 1 - m$ dimensional boundary C_m of this hypersurface S_m is critical. Whereas the form of the hypersurfaces S_m and C_m essentially depends on the regularization scheme of the theory, the m-critical behaviour on the hypersurface C_m is universal and depends only on m. For the Sine-Gordon potential (essentially an infinite degree polynomial interaction), $m = 2, 3, 4, \ldots, \infty$-critical behaviours are possible and our observation is that by turning on full set of coupling constants we can fully explore the ∞-critical behaviour at IR.

In string theory language, the m-critical behaviour in the IR is nothing but a free $c = 1$ boundary conformal field theory of m-coalescing D24-branes with $U(m)$ Chan-Paton factor. Note that in order to probe the ∞-critical behaviour (in other words $U(\infty)$ symmetry of the nonperturbative bosonic vacuum reached asymptotically from the core of the D24-brane) we need to turn on all tachyonic modes even though higher modes, being highly irrelevant, have no significant effect in perturbative RG analysis. When we perturb by the zeroth mode only (where $m = 0$) we end up with the closed string vacuum with no D-brane (*i.e.* $g_{IR}/g_{UV} \to 0$).

At this point we can draw an interesting analogy from Kondo physics[3]. The Kondo model describes the interaction between k-degenerate bands of spin-$\frac{1}{2}$ conduction electrons and a quantum impurity of arbitrary size s placed at one boundary, say at origin, which essentially makes the system nontranslationally invariant. This type of problem is similar to the open string the-

[3]For a treatment on CFT description of it see [40].

ory in the sense that boundary spin is like a dynamic degrees of freedom or Chan-Paton factor. The IR (or low-temperature) behaviour is quite different depending on the relative size of $2s$ and k. For the *underscreened case* (*i.e.* $s > \frac{k}{2}$) the RG flow leads to IR fixed point with Fermi liquid behaviour. In the course of RG flow the boundary spin fuses with internal spin of the system and the electron channel screens or swallows up a part of the impurity spin giving rise to a similar current algebra as that of the UV fixed point with a shifted Kac-Moody current,

$$\overrightarrow{\mathcal{J}}_n = \overrightarrow{J}_n + \overrightarrow{S}\,, \tag{5.46}$$

leading to rearrangement in the conformal towers of the theory [4]. The fact that the IR fixed point for the underscreened case is stable can be seen from the interaction between the partially screened spin and the electrons on the neighbouring lattice sites, which is ferromagnetic and hence irrelevant. The strong coupling fixed point is much the same as the UV or zero coupling fixed point. Only the size of the impurity spin is reduced and there is a change in boundary condition on the otherwise free fermions (corresponding to a $\pi/2$ phase shift). This is precisely what happens in the present context of tachyon condensation which leads to a change of boundary condition from Neumann to Dirichlet. The leftover impurity spin $s-\frac{k}{2}$ at IR decouples from the internal symmetry group and is similar to the $U(k)$ Chan-Paton factor of k D24-branes at IR.

For *exact screening*, $s = \frac{k}{2}$, the internal spin fully swallows up the boundary spin giving rise to a translationally invariant system which is analogous to translationally invariant nonperturbative closed string vacuum with no D-brane.

For the *overscreened* case (*i.e.* $s < \frac{k}{2}$), the IR fixed point is very nontrivial and is determined by fusion with the spin-s primary. The theory arrives at non-Fermi liquid fixed point in the IR which describes completely different physics. The strong coupling IR fixed point ground state has an overscreened spin of size $\frac{k}{2} - s$. The induced interaction is antiferromagnetic and hence unstable. The existence of such nontrivial fixed point can be proved in the large-k limit (keeping s fixed). The ground state cannot be described by a simple physical picture. In [42] the overscreened case is described based on hamiltonian non-abelian bosonization. The idea is to represent the $2k$ species of electrons in terms of three independent bosonic fields: a free scalar carrying $U(1)$ charge, a $SU_k(2)$ WZW non-linear σ-model field carring $SU(2)$ spin, and a $SU_2(k)$ WZW field carring $SU(k)$ flavour degrees of freedom. The Kondo interaction involves only the spin current. The degrees of freedom at the IR fixed point are described by an $U(1) \times SU_k(2) \times SU_2(k)$ Kac-Moody invariant CFT. Both boundary scaling dimensions

[4] For exactly screened and underscreened cases, $s \geq \frac{k}{2}$, the IR fixed point is given by fusion with spin-$\frac{k}{2}$ priamry. The fusion rules are [41] $j \otimes \frac{k}{2} = \frac{k}{2} - j$. Each conformal tower is mapped into a unique conformal tower giving rise to free fermion spectrum with a $\pi/2$ phase shift. Whereas IR fixed point in overscreened case is given by fusion with spin-s primary.

and Fermi-liquid bulk scaling dimensions are described by a $U(1) \times SU_k(2) \times SU_2(k)$ Kac-Moody boundary CFT. There are restrictions on the combinations of charge, spin, flavour degrees of freedom. Unlike the underscreened case, here fermion exponents are not recovered and are generally not half-integer. They come from more general combinations of three types of bosonic fields. The fermion exponents are sums of $U(1)$, $SU_2(k)$ and $SU_k(2)$ Kac-Moody current algebra exponents. They are not free and in fact *bound* together in combinations to form fermion composites.

The analogue of overscreened Kondo physics in string theory is the case when one considers string theory on the $SU(2)$ group manifold that can be described by a world-sheet WZW action and D-branes are stabilized against shrinking by quantized $U(1)$ flux as discussed by Bachas, Douglas and Schweigert [43] (also see [44]). They considered a static D2-brane wrapping an S^2 parametrized by (θ, ϕ) breaking $SU(2)_L \times SU(2)_R$ to the diagonal $SU(2)_{diag}$. With non-zero background B field the gauge invariant quantity is $\mathcal{F} = B + 2\pi F$. If $0 < n < k$, where n is the magnetic monopole number appearing due to the world-volume flux quantization $\int F$, the D2 brane is prevented from shrinking to one of the poles of S^3 [43]. Inside this range the energy of the system has a unique minimum away from the poles of S^3. In the large-k limit, that minimum energy reduces to the mass of n D-particles. The stable configuration leads in the dual picture to a bound state of n D-particles on S^2 similar to the case described in [45].

From the above physical picture (of the underscreening and exact screening Kondo effect) one can try to see the mechanism of generation of local $U(m)$ Chan-Paton factors of m coincident D24-branes as a result of breaking of inherent stringy $U(\infty)$ symmetry due to fusion of boundary spin or dynamical degrees of freedom with internal symmetry. The local operator $\mathcal{O} = g_m \Phi^{2m}(0)$ near the IR fixed point breaks $U(\infty)$ down to $U(\infty - m) \times U(m)$ in the core of the soliton. While there is a local $U(m)$ on m coincident D24-branes, $U(\infty - m)$ is again swallowed up by the internal symmetry of the system. As $m \to 0$ (the exact screening case), the situation is similar to perturbation by the tachyon zero mode only, which leads the theory to the nonperturbative closed string vacuum and the local $U(1)$ symmetry on the D25-brane is restored to the full $U(\infty)$ symmetry. For $m \to \infty$ (when perturbations with all λ_is are triggered) we get full $U(\infty)$ symmetry corresponding to infinite number of D24-branes in the core of the soliton. Symmetry restoration in the same context for noncommutative tachyon solitons is discussed in [46].

It can also be seen whether the operator $\mathcal{O} = g_m \Phi^{2m}(0)$ can give rise to m^2 copies of identity operator at IR fixed point, which are basically m^2 generators of $U(m)$ as all operators flow to identity operators or derivative of identity operators at IR fixed point. So from the multicritical behaviour discussed above, four copies of the identity operator are left, indicating the final configuration to be that of two D24-branes.

6 Final remarks

There are many unexplored questions related to the material covered in this paper. We mention some of them below along with some remarks.

It would be interesting to understand the precise relationship between boundary string field theory and Chern-Simons string field theory. In other words, how the action in boundary SFT is related to the Chern-Simons action in cubic SFT. It is pointed out in [27] that the boundary SFT action described in certain coordinates on the space of coupling constants must be related to the cubic action described in a particular choice of coordinates on the space of string fields. The two choices of coordinates are related by some complicated singular transformation. In [27] the boundary conformal field theory is perturbed by a simple tachyon profile with mass parameter u that flows from zero in the UV to infinity in the IR. The lump profile has width $1/u$ which vanishes in the IR. In contrast, the level truncation scheme in cubic SFT gives solitons with finite width[5].

On the other hand, in our scheme one gets the soliton profile with finite width situated in a nearby IR fixed point. The set of values the first three tachyon harmonics evolve to in order to hit the desired IR fixed point is very close to the one obtained in the level truncation scheme in cubic SFT [11]. Also the perturbative result of the boundary entropy is in good agreement with the exact result. It might be more appropriate to choose the signs of all the couplings initially to be the same or by a symmetry to be of the same sign for the zero and second modes and of opposite sign for the first mode. The former choice does not make the analysis different - we have already mentioned that the zero mode is the identity operator which appears in the RG equations only in linear order and does not contribute to the boundary entropy (which is the only physical quantity of interest here). The latter choice (that involves opposite signs for first and second modes) makes the calculation of the boundary entropy more complicated and it does not fit in well with the setup used in [29]. While the relationship between the boundary SFT and the Chern-Simons SFT is not clear, finding the reason for this (apparent) agreement of our result with that of [11] is left for future work.

There is a striking similarity between the beta function equations (3.26) in the first quantized theory and the string field equations of motion given below (in the notation of [11]):

$$\begin{aligned}
2\frac{\partial V}{\partial t_0} &= 2t_0 - 2K^3 t_0^2 - K^{3-\frac{2}{R^2}} t_1^2 - K^{3-\frac{2.2^2}{R^2}} t_2^2 = 0\,, \\
2\frac{\partial V}{\partial t_1} &= -\Big(1-\frac{1}{R^2}\Big)t_1 + 2K^{3-\frac{2}{R^2}} t_0 t_1 + K^{3-\frac{2.3}{R^2}} t_1 t_2 = 0\,,
\end{aligned}$$

[5]This in fact is not a disagreement since in the corresponding Zamolodchikov metric on the field space the distance between the perturbative UV fixed point and the stable minimum is finite.

$$2\frac{\partial V}{\partial t_2} \;=\; -\Big(1-\frac{4}{R^2}\Big)t_2 + 2K^{3-\frac{2.4}{R^2}}t_0t_2 + K^{3-\frac{2.3}{R^2}}t_1^2 = 0\,, \tag{6.47}$$

where $K = \frac{3\sqrt{3}}{4}$ is the inverse of the mapping radius of the punctured disks defining the three string vertex. It is maximal for the vertex of the cubic potential [47]. The main difference is in the zero mode dependence. In the RG case the identity operator adds a constant to the action. In the RG equation the corresponding coupling constant appears in the universal linear order only. Presumably there are definite complicated nonlinear relations between the different tachyon harmonics generating term proportional to t_0^2 and the cross terms t_0t_1, t_0t_2 in RG equations. On the other hand since $\tilde{\lambda} = a\lambda$ is dimensionless, the mapping radius K^{-1} should be somehow related to the cut-off $a \sim \Lambda^{-1}$ in boundary conformal field theory side.

We have not raised issues concerning the shape and size of the lump representing the lower dimensional D-brane. As was pointed out in [11], in order to get some insight into this issue one needs to find the energy density of the object. Since cubic SFT is nonlocal, it is presently not clear how to address this. Also the size might be an artifact of the particular gauge chosen. The shape and size depends on a particular definition of off-shell string field, and can vary from one SFT to another. The important fact is that in Chern-Simons SFT everything is smooth and non-singular, so it gives a good definition of the off-shell string field [6].

There is no invariant definition of size of the D-brane unless one looks at the coupling to metric or some closed string background. Presumably one needs to follow the analysis analogous to the bulk conformal perturbation theory in the presence of a fluctuating metric (see for example [48, 49]). It is pointed out in [50] that the dynamics of the metric makes sense only when there is a scale in the path integral, for example by fixing the area. This is done by the so-called *gravitational dressing* of the microscopic matter operator by the appropriate Liouville dependence and inserting it in the path integral. An analogous treatment in boundary perturbation theory might give some useful insight regarding the shape and size of the D-brane.

An other important issue is to extend the scheme used in this paper to superstring theories where lower dimensional D-branes are identified with a tachyonic kink solution rather than a lump solution[7]. In the supersymmetric setup it is more convenient and sensible to study a configuration of parallel D-branes. Also the size and shape of the D-branes in the IR can be investigated considering physical picture involving closed string scattering off the D-branes [52]. It would also be of interest to study intersecting D-brane configurations using the boundary RG analysis discussed in this paper.

[6]On the other hand due to UV divergences on the worldsheet it is difficult to define a general off-shell amplitude in boundary SFT and one needs to follow an appropriate regularization scheme.

[7]Very recently superstring generalization of the results of [27] has appeared [51].

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A Appendix

In this appendix we will present the details of the RG calculation in order two and three.

Second order contribution:

Using our ansatz for Green's function of fast moving modes (3.20), the $O(\lambda_1^2)$ part of (3.17) turns out to be

$$\begin{aligned}\frac{1}{2}(\langle S_I\rangle_f^2 - \langle S_I^2\rangle_f) &= -\frac{\lambda_1^2}{4}\int d\tilde{R}\int dr\Big[\Big\{-\frac{\beta^2}{2\pi}dl\Big(1+ln\Big|\frac{a}{r}\Big|\Big)\Big\}\cos 2\beta[\Phi_{s\Lambda'}(\tilde{R})] \\ &\quad +\Big\{\frac{\beta^2}{2\pi}dl\Big(1+ln\Big|\frac{a}{r}\Big|\Big)\Big\}\cos\beta[r\partial_{\tilde{R}}\Phi_{s\Lambda'}(\tilde{R})]\Big] + \dots \end{aligned} \tag{A.1}$$

where we have expanded the cosines in r. The ellipsis represent the $O(\lambda_2^2)$ (which is identical to the $O(\lambda_1^2)$ part under replacement of β by 2β) and $O(\lambda_1\lambda_2)$ contributions. Following the arguments given in section 2, we reach to the following expression

$$\begin{aligned}\frac{1}{2}(\langle S_I\rangle_f^2 - \langle S_I^2\rangle_f) &= \frac{\lambda_1^2}{4}\Big[\int dr\Big\{\frac{\beta^2}{2\pi}dl\Big(1+ln\Big|\frac{a}{r}\Big|\Big)\Big\}\Big]\int d\tilde{R}\ \cos 2\beta[\Phi_{s\Lambda'}(\tilde{R})] \\ &\quad -\frac{\lambda_1^2}{4}\Big[\int dr\Big\{\frac{\beta^2}{2\pi}dl\Big(1+ln\Big|\frac{a}{r}\Big|\Big)\Big\}\Big]\int d\tilde{R}\ \mathbf{1} + \dots , \end{aligned} \tag{A.2}$$

where the integration over r gives rise to nonuniversal numerical constant I_1 given by

$$I_1 = \frac{1}{8\pi}\int_{-a}^{a} dr\ \left(1+ln\Big|\frac{a}{r}\Big|\right) = \frac{a}{2\pi}\,. \tag{A.3}$$

Similarly use of center-of-mass and relative coordinates $\tilde{R}$ and r respectively and expansion of the cosines in r simplifies the $O(\lambda_1\lambda_2)$ contribution to (3.17) into

$$
\begin{aligned}
&\frac{1}{2}(\langle S_I\rangle_f^2-\langle S_I^2\rangle_f)\\
&=\lambda_1\lambda_2\beta^2\Big(1-\frac{5\beta^2}{4\pi}dl\Big)\int dr\ G_f(r)\int d\tilde{R}\ \Big(\cos\beta[3\Phi_{s\Lambda'}(\tilde{R})+\frac{r}{2}\partial_{\tilde{R}}\Phi_{s\Lambda'}(\tilde{R})]\\
&-\cos\beta[\Phi_{s\Lambda'}(\tilde{R})+\frac{3r}{2}\partial_{\tilde{R}}\Phi_{s\Lambda'}(\tilde{R})]\Big)+\ldots\\
&=\lambda_1\lambda_2\beta^2\Big(1-\frac{5\beta^2}{4\pi}dl\Big)\int dr\ G_f(r)\int d\tilde{R}\ \Big[\cos 3\beta\Phi_{s\Lambda'}(\tilde{R})\Big(1-\frac{\beta^2}{2}\ \frac{r^2}{4}(\partial_{\tilde{R}}\Phi_{s\Lambda'}(\tilde{R}))^2\Big)\\
&-\sin 3\beta\Phi_{s\Lambda'}(\tilde{R})\cdot\frac{r}{2}\partial_{\tilde{R}}\Phi_{s\Lambda'}(\tilde{R})-\cos\beta\Phi_{s\Lambda'}(\tilde{R})\Big(1-\frac{\beta^2}{2}\ \frac{9r^2}{4}(\partial_{\tilde{R}}\Phi_{s\Lambda'}(\tilde{R}))^2\Big)\\
&+\sin\beta\Phi_{s\Lambda'}(\tilde{R})\cdot\frac{3r}{2}\partial_{\tilde{R}}\Phi_{s\Lambda'}(\tilde{R})\Big]+\ldots\,, \qquad\qquad (A.4)
\end{aligned}
$$

where in the last step we expanded the cosines in r about $\tilde{R}$. By the same argument applied to $O(\lambda_1^2)$ part we can neglect the (gradient)^2 terms. As the field Φ is odd under $\tilde{R}$ reflection, the overall sine terms are not invariant under this operation and hence do not contribute. Hence using (3.20) and the nonuniversal constants resulting from the integrals

$$
\begin{aligned}
I_2 &= -\frac{1}{48\pi}\int_{-a}^{a}dr_1\int_{-a}^{a}dr_2\ ln\left|\frac{(r_1-r_2)r_2}{r_1^2}\right|=\frac{\pi a^2}{144}\,,\\
I_3 &= -\frac{1}{48\pi}\int_{-a}^{a}dr_1\int_{-a}^{a}dr_2\ ln\left|\frac{(r_1-r_2)r_2^3}{r_1^4}\right|=\frac{\pi a^2}{144}\,, \qquad (A.5)
\end{aligned}
$$

the $O(\lambda_1\lambda_2)$ contribution simplifies to

$$
\frac{1}{2}(\langle S_I\rangle_f^2-\langle S_I^2\rangle_f)=2\frac{\tilde{\lambda}_1\tilde{\lambda}_2}{\pi}\beta^2 dl\int\frac{d\tilde{R}}{a}\ (\cos 3\beta\Phi_{s\Lambda'}(\tilde{R})-\cos\beta\Phi_{s\Lambda'}(\tilde{R}))+\ldots\,, \qquad (A.6)
$$

neglecting the $O(dl^2)$ terms. Typically the integrals in (A.5) are divergent. But we have regulated them excluding the origin which is sensible in present case.

Third order contribution:

The cubic contribution to (3.7) reads

$$
\begin{aligned}
&\frac{1}{6}\langle S_I^3\rangle_f-\frac{1}{2}\langle S_I\rangle_f\langle S_I^2\rangle_f+\frac{1}{3}\langle S_I\rangle_f^3\\
=&\ -\frac{\lambda_1^3}{6}\int dt_1\int dt_2\int dt_3\Big\{\Big(\frac{\beta^2}{8\pi}dl\ ln\left|\frac{(t_2-t_3)(t_3-t_1)}{(t_1-t_2)^2}\right|\Big)\cos\beta[\Phi_{s\Lambda'}(t_1)+\Phi_{s\Lambda'}(t_2)+\Phi_{s\Lambda'}(t_3)]
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{\beta^2}{8\pi}dl\ \left(ln\left|\frac{a^4}{(t_1-t_2)^2(t_2-t_3)(t_3-t_1)}\right|+4\right)\right)\cos\beta[\Phi_{s\Lambda'}(t_1)+\Phi_{s\Lambda'}(t_2)-\Phi_{s\Lambda'}(t_3)]\\
&+\left(\frac{\beta^2}{8\pi}dl\ \left(ln\left|\frac{(t_1-t_2)^2(t_3-t_1)}{a^2(t_2-t_3)}\right|-2\right)\right)\cos\beta[\Phi_{s\Lambda'}(t_1)-\Phi_{s\Lambda'}(t_2)+\Phi_{s\Lambda'}(t_3)]\\
&+\left(\frac{\beta^2}{8\pi}dl\ \left(ln\left|\frac{(t_1-t_2)^2(t_2-t_3)}{a^2(t_3-t_1)}\right|-2\right)\right)\cos\beta[\Phi_{s\Lambda'}(t_1)-\Phi_{s\Lambda'}(t_2)-\Phi_{s\Lambda'}(t_3)]\Bigg\}+\dots,
\end{aligned}
\tag{A.7}
$$

neglecting the $O(dl^2)$ terms. The ellipsis represent the $O(\lambda_2^3)$, $O(\lambda_1^2\lambda_2)$, and $O(\lambda_1\lambda_2^2)$ contributions. The $O(\lambda_2^3)$ and $O(\lambda_1\lambda_2^2)$ are related to $O(\lambda_1^3)$ and $O(\lambda_1^2\lambda_2)$ by exchanging $\beta\leftrightarrow 2\beta$ respectively. We again introduce center-of-mass coordinate $\bar{R}=(t_1+t_2+t_3)/3$ and relative coordinates $r_1=t_2-t_1$ and $r_2=t_3-t_1$. The relative coordinates being bounded within the UV cut-off $r_1,r_2<\Lambda^{-1}\sim a$, the Green's functions are truly short-ranged and we can expand the cosines in r_1,r_2. The result is

$$
\begin{aligned}
&\frac{1}{6}\langle S_I^3\rangle_f-\frac{1}{2}\langle S_I\rangle_f\langle S_I^2\rangle_f+\frac{1}{3}\langle S_I\rangle_f^3\\
=\ &-\lambda_1^3\frac{\beta^2}{48\pi}dl\Bigg[\int dr_1\int dr_2\ ln\Big|\frac{(r_1-r_2)r_2}{r_1^2}\Big|\int d\bar{R}\ \cos 3\beta\Phi_{s\Lambda'}(\bar{R})\\
&+\int dr_1\int dr_2\ (4+4\ ln|a|-ln|r_1^2r_2(r_1-r_2)|)\\
&\cdot\int d\bar{R}\ \cos\beta[\Phi_{s\Lambda'}(\bar{R})-\frac{4}{3}r_1\partial_{\bar{R}}\Phi(\bar{R},r_2)+\frac{2}{3}r_2\partial_{\bar{R}}\Phi_{s\Lambda'}(\bar{R},r_1)]\\
&+\int dr_1\int dr_2\ (-4-4\ ln|a|+2\ ln\Big|\frac{r_1^2(r_1-r_2)}{r_2}\Big|)\\
&\cdot\int d\bar{R}\ \cos\beta[\Phi_{s\Lambda'}(\bar{R})-\frac{4}{3}r_2\partial_{\bar{R}}\Phi_{s\Lambda'}(\bar{R},r_1)+\frac{2}{3}r_1\partial_{\bar{R}}\Phi_{s\Lambda'}(\bar{R},r_2)]\Bigg]+\dots.
\end{aligned}
\tag{A.8}
$$

Notice that the expression

$$
\begin{aligned}
&\int dr_1\int dr_2\ (4+4\ ln|a|)\cos\beta[\Phi_{s\Lambda'}(\bar{R})-\frac{4}{3}r_1\partial_{\bar{R}}\Phi_{s\Lambda'}(\bar{R},r_2)+\frac{2}{3}r_2\partial_{\bar{R}}\Phi_{s\Lambda'}(\bar{R},r_1)]\\
&+\int dr_1\int dr_2\ (-4-4\ ln|a|)\cos\beta[\Phi_{s\Lambda'}(\bar{R})-\frac{4}{3}r_2\partial_{\bar{R}}\Phi_{s\Lambda'}(\bar{R},r_1)+\frac{2}{3}r_1\partial_{\bar{R}}\Phi_{s\Lambda'}(\bar{R},r_2)]
\end{aligned}
\tag{A.9}
$$

vanishes identically under the exchange of the dummy variables r_1,r_2. Hence the cubic contribution simplifies to

$$
\frac{1}{6}\langle S_I^3\rangle_f-\frac{1}{2}\langle S_I\rangle_f\langle S_I^2\rangle_f+\frac{1}{3}\langle S_I\rangle_f^3
$$

$$
\begin{aligned}
= & -\lambda_1^3 \frac{\beta^2}{48\pi} dl \Big[\int dr_1 \int dr_2 \; ln\Big|\frac{(r_1-r_2)r_2}{r_1^2}\Big| \int d\bar{R} \;\; \cos 3\beta\Phi_{s\Lambda'}(\bar{R}) + \int dr_1 \int dr_2 \; ln\Big|\frac{r_2^3(r_1-r_2)}{r_1^4}\Big| \\
& \cdot \int d\bar{R} \;\; \cos\beta[\Phi_{s\Lambda'}(\bar{R}) - \frac{4}{3} r_1 \partial_{\bar{R}} \Phi_{s\Lambda'}(\bar{R}, r_2) + \frac{2}{3} r_2 \partial_{\bar{R}} \Phi_{s\Lambda'}(\bar{R}, r_1)] \Big] + \ldots . \qquad \text{(A.10)}
\end{aligned}
$$

Since the relative coordinates r_1, r_2 are small, the second cosine simplifies to

$$
\begin{aligned}
& \cos\beta[\Phi_s(\bar{R}) - \frac{4}{3} r_1 \partial_{\bar{R}} \Phi_s(\bar{R}, r_2) + \frac{2}{3} r_2 \partial_{\bar{R}} \Phi_s(\bar{R}, r_1)] \\
\approx & \cos\beta\Phi_s(\bar{R}) \Big(1 - \frac{\beta^2}{2} \Big(\frac{2}{3} r_2 \partial_{\bar{R}} \Phi_s(\bar{R}, r_1) - \frac{4}{3} r_1 \partial_{\bar{R}} \Phi_s(\bar{R}, r_2) \Big)^2 \Big) \\
& - \Big(\frac{2}{3} r_2 \partial_{\bar{R}} \Phi_s(\bar{R}, r_1) - \frac{4}{3} r_1 \partial_{\bar{R}} \Phi_s(\bar{R}, r_2) \Big) \sin\beta\Phi_s(\bar{R}) \, .
\end{aligned}
\qquad \text{(A.11)}
$$

Substituting this into (A.8) we get

$$
\begin{aligned}
& \frac{1}{6}\langle S_I^3 \rangle_f - \frac{1}{2} \langle S_I \rangle_f \langle S_I^2 \rangle_f + \frac{1}{3} \langle S_I \rangle_f^3 \\
= & -\lambda_1^3 \frac{\beta^2}{48\pi} dl \Big[\int dr_1 \int dr_2 \; ln\Big|\frac{(r_1-r_2)r_2}{r_1^2}\Big| \int d\bar{R} \;\; \cos 3\beta\Phi_{s\Lambda'}(\bar{R}) \\
& + \int dr_1 \int dr_2 \; ln\Big|\frac{r_2^3(r_1-r_2)}{r_1^4}\Big| \int d\bar{R} \;\; \cos\beta\Phi_{s\Lambda'}(\bar{R}) \\
& - \frac{\beta^2}{2} \int dr_1 \int dr_2 \; ln\Big|\frac{r_2^3(r_1-r_2)}{r_1^4}\Big| \Big(\frac{4}{9} r_2^2 + \frac{16}{9} r_1^2 - \frac{16}{9} r_1 r_2 \Big) \int d\bar{R} \;\; \cos\beta\Phi_{s\Lambda'}(\bar{R}) \Big(\partial_{\bar{R}} \Phi_{s\Lambda'}(\bar{R}) \Big)^2 \\
& - \int dr_1 \int dr_2 \int d\bar{R} \; ln\Big|\frac{r_2^3(r_1-r_2)}{r_1^4}\Big| \Big(\frac{2}{3} r_2 \partial_{\bar{R}} \Phi_{s\Lambda'}(\bar{R}, r_1) - \frac{4}{3} r_1 \partial_{\bar{R}} \Phi_{s\Lambda'}(\bar{R}, r_2) \Big) \sin\beta\Phi_{s\Lambda'}(\bar{R}) \Big] + \ldots .
\end{aligned}
\qquad \text{(A.12)}
$$

Since under reflection of $\bar{R}$ the integrand containing sine is odd, it vanishes identically. The $\bar{R}$ integration containing $\cos\beta\Phi(\bar{R})\Big(\partial_{\bar{R}}\Phi(\bar{R})\Big)^2$ vanishes imposing equation of motion from the action where the bulk degrees of freedom have been integrated out. Hence the expression for $O(\lambda_1^3)$ cubic contribution is

$$
\begin{aligned}
& \frac{1}{6}\langle S_I^3 \rangle_f - \frac{1}{2} \langle S_I \rangle_f \langle S_I^2 \rangle_f + \frac{1}{3} \langle S_I \rangle_f^3 \\
= & -\lambda_1^3 \frac{\beta^2}{48\pi} dl \Big[\int dr_1 \int dr_2 \; ln\Big|\frac{(r_1-r_2)r_2}{r_1^2}\Big| \int d\bar{R} \;\; \cos 3\beta\Phi_{s\Lambda'}(\bar{R}) \\
& + \int dr_1 \int dr_2 \; ln\Big|\frac{r_2^3(r_1-r_2)}{r_1^4}\Big| \int d\bar{R} \;\; \cos\beta\Phi_{s\Lambda'}(\bar{R}) \Big] + \ldots , \qquad \text{(A.13)}
\end{aligned}
$$

where the integrations over r_1, r_2 give rise to nonuniversal constants given by (A.5).

Next we turn to $O(\lambda_1^2\lambda_2)$ part. Using (3.10), (3.15) and (3.20) and neglecting $O(dl^2)$ terms it simplifies to

$$\begin{aligned}
&\frac{1}{6}\langle S_I^3\rangle_f - \frac{1}{2}\langle S_I\rangle_f\langle S_I^2\rangle_f + \frac{1}{3}\langle S_I\rangle_f^3 \\
= \;&\frac{1}{8}\lambda_1^2\lambda_2\Big(1-\frac{3\beta^2}{2\pi}dl\Big)\int dt_1\int dt_2\int dt_3\Bigg[\frac{\beta^2}{\pi}dl\; ln\Big|\frac{t_3-t_1}{t_3-t_2}\Big|\cos\beta(\Phi_{s\Lambda'}(t_1)+\Phi_{s\Lambda'}(t_2)+2\Phi_{s\Lambda'}(t_3)) \\
&-\frac{\beta^2}{\pi}dl\; ln\Big|\frac{t_3-t_1}{t_3-t_2}\Big|\cos\beta(-\Phi_{s\Lambda'}(t_1)-\Phi_{s\Lambda'}(t_2)+2\Phi_{s\Lambda'}(t_3)) \\
&-\bigg(\frac{\beta^2}{\pi}dl\; ln\Big|\frac{a^2}{(t_3-t_2)(t_3-t_1)}\Big|+\frac{2\beta^2}{\pi}dl\bigg)\cos\beta(\Phi_{s\Lambda'}(t_1)-\Phi_{s\Lambda'}(t_2)+2\Phi_{s\Lambda'}(t_3)) \\
&+\bigg(\frac{\beta^2}{\pi}dl\; ln\Big|\frac{a^2}{(t_3-t_2)(t_3-t_1)}\Big|+\frac{2\beta^2}{\pi}dl\bigg)\cos\beta(-\Phi_{s\Lambda'}(t_1)+\Phi_{s\Lambda'}(t_2)+2\Phi_{s\Lambda'}(t_3))\Bigg]+\ldots
\end{aligned} \tag{A.14}$$

As in the previous case we introduce the center-of-mass coordinate $\bar{R} = (t_1 + t_2 + t_3)/3$ and relative coordinates $r_1 = t_3 - t_1$ and $r_2 = t_3 - t_2$. Expanding the cosines in r_1, r_2, the above expression can be written approximately as

$$\frac{1}{8}\frac{\beta^2}{\pi}dl\int dr_1\int dr_2\;(ln\;r_1 - ln\;r_2)\cdot\int d\bar{R}(\cos 4\beta\Phi_{s\Lambda'}(\bar{R})-\mathbf{1}) \tag{A.15}$$

which vanishes identically after integration over r_1 and r_2. By doing similar analysis for $O(\lambda_1\lambda_2^2)$, we conclude that all the contributions to $O(\lambda_1^2\lambda_2)$ and $O(\lambda_1\lambda_2^2)$ vanishes.

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